# CHAPTER 9 <br> SHELLS AND STRUCTURES 

## DRAFT

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### 9.1 INTRODUCTION

Shell elements and other structural elements are invaluable in the modeling of many engineered components and natural structures. Thin shells appear in many products, such as the sheet metal in an automobile, the fuselage, wings and rudder of an airplane, the housings of products such as cell phones, washing machines, computers. Modeling these items with continuum elements would require a huge number of elements and lead to extremely expensive computations. As we have seen in Chapter 8, modeling a beam with hexahedral continuum elements requires a minimum of about 5 elements through the thickness. Thus even a low order shell element can replace 5 or more continuum elements, which improves computational efficiency immensely. Furthermore, modeling thin structures with continuum elements often leads to elements with high aspect ratios, which degrades the conditioning of the equations and the accuracy of the solution. In explicit methods, continuum element models of shells are restricited to very small stable time steps. Thus it can be seen that structural elements are very useful in engineering analysis.

Structural elements are classified as:

1. beams, in which the motion is described as the function of a single independent variable;
2. shells, where the motion is described as a function of two independent variables;
3. plates, which are flat shells.

Plates are usually modeled by shell elements in computer software. Since they are just flat shells, we will not consider plate elements separately. Beams on the other hand, require some additional theoretical considerations and provide simple models for learning the fundamentals of structural elements, so we will devote a substantial part of this chapter to beams.

There are two approaches to developing shell finite elements:

1. develop the formulation for shell elements by using classical straindisplacement and momentum (or equilibrium) equations for shells to develop a weak form of the momentum (or equilibrium) equations;
2. develop the element directly from a continuum element by imposing the structural assumptions on the weak form or on the discrete equation; this is called the continuum based (CB) approach.
The first approach is difficult, particularly for nonlinear shells, since the governing equations for nonlinear shells are very complex and awkward to deal with; they are usually formulated in terms of curvilinear components of tensors, and features such as
variations in thickness, junctions and stiffeners are generally difficult to incorporate. There is still disagreement as to what are the best nonlinear classical shell equations. The CB (continuum-based) approach, on the other hand, is straightforward, yields excellent results, is applicable to arbitrarily llarge deformations and is widely used in commercial software and research. Therefore we will concentrate on the CB methodology. It is also called the degenerated continuum approach; we prefer the appellation continuum based, coined by Stanley(1985), since there is nothing degenerate about it.

The CB methodology is not only simpler, but intellectually a more appealing appraoch than classical shell theories for developing shell elements. In most plate and shell theories, the equilibrium or momentum equations are developed by imposing the structural assumptions on the motion and then using the principle of virtual work to develop the partial differential equations for momentum balance or equilibrium. The development of a weak form for these shell momentum equations than entails going back to the principle of virtual work. In the CB approach, the kinematic assumptions are either

1. imposed on the motion in the weak form of the momentum equations for continua or
2. imposed directly on the discrete equations for continua.

Thus the CB shell formulation is a more straightforward way of obtaining the discrete equations for shells and structures.

We will begin with a description of beams in two dimensions. This will provide a setting for clearly and easily describing the assumptions of various structural theories and comparing them with CB beam elements. In contrast to the schema in previous Chapters, we will begin with the implementation, for in the implementation the simplicity and key features of the CB approach are most transparent. We will then examine CB beam elements more thoroughly from a theoretical viewpoint.

The CB approach is subsequently employed for the development of shell elements. Again, we begin with the implementation, illustrating how many of the techniques developed for continuum elements in the previous chapters can be applied directly to shells. The CB shell theory developed here is a synthesis of various approaches reported in the literature but also incorporates a new treatment of changes in thickness due to large deformations and conservation of matter. As part of this treatment, the methodologies for describing large rotations in three dimensions are described.

Two of the pitfalls of CB shell elements are then examined: shear and membrane locking. These phenomena are examined in the context of beams but the insights gained are applicable to shell elements. Methods for circumventing these difficulties by means of assumed strain fields are described and examples of elements which alleviate shear and membrane locking are given.

We conclude with a description of 4-node quadrilateral shell elements that evaluate the internal nodal forces with one stack of quadrature points, often called onepoint quadrature elements. These elements are widely used in explicit methods and large scale analysis. Several elements of this genre are reviewed and compared and the techniques for consistently controlling the hourglass modes which result from the underintegration are described.

### 9.2 TWO DIMENSIONAL BEAMS

9.2.1. Governing Equations and Assumptions. In this Section the CB theory is developed for beams. In addition, we develop a beam element based on classical beam theory.

The governing equations for structures are identical to those for continua:

1. conservation of matter
2. conservation of linear and angular momentum
3. conservation of energy
4. constitutive equations
5. strain-displacement equations

The key feature which distinguishes structures from continua is that assumptions are made about the motion and the state of stress in the element. In other words, the motion is constrained so that it satisfies certain hypothesis which are based on experimental observations on the motion of thin structures and shells. The assumptions on the motion are called kinematic assumptions, the assumptions on the stress field are called kinetic assumptions.

The major kinematic assumption concerns the motion of the normals to the midline (also called reference line) of the beam. In linear structural theory, the midline is usually chosen to be the loci of the centroids of the cross-sections of the beam. However, the selection of a reference line has no effect on the response of a CB element: any line which corresponds approximately to the shape of the beam may be chosen as the reference line. The choice of reference line only effects the values of the resultant moments; the stresses and the overall response are not affected. We will use the terms reference line and midline interchangeably, noting that even when the term midline is used the precise location of this line relative to the cross-section of the beam is irrelevant in a CB element. The plane defined by the normals to the midline is called the normal plane. Fig. 9.2 shows the reference line and normal plane for a beam.


Figure 9.2. Motion in an Euler-Bernoulli bean and a shear (Mindlin-Reissner) beam; in the Euler-Bernoulli beam, the normal plane remains plane and normal, whereas in the shear beam the normal plane remains plane but not normal.

Two types of beam theory are widely used: Euler-Bernoulli beam theory and shear beam theory. The kinematic assumptions of these theories are:

1. in Euler-Bernoulli beam theory the planes normal to the midline are assumed to remain plane and normal; this is also called engineering beam theory while the corresponding shell theory is called the Kirchhoff-Love shell theory;
2. in shear beam theory the planes normal to the midline are assumed to remain plane; this is also called Timoshenko beam theory, and the corresponding shell theory is called the Mindlin-Reissner shell theory;
Euler-Bernoulli beams, as we shall see shortly, do not admit any transverse shear, whereas beams governed by the second assumption do admit transverse shear. The motions of an Euler-Bernoulli beam are a subset of the motions encompassed by shear beam theory.

For the purpose of describing the consequences of these kinematic assumptions, we consider a straight beam along the $x$-axis in two dimensions as shown in Fig. 9.2. Let the $x$-axis coincide with the midline and the $y$-axis with the normal to the midline. We consider only the instant when the beam is in the configuration described, so the following equations do not constitute a nonlinear theory. We will first express the kinematic assumptions mathematically and develop the rate-of-deformation tensor; the rate-of-deformation will have the same properties as the linear strain since the equations for the rate-of-deformation can be obtained by replacing velocities by displacements in the linear strain equations. The aim of the following is to illustrate the consequences of the kinematic assumptions on the strain field, not to construct a theory which is worth implementing.
9.9.2. Timoshenko (Shear Beam) Theory. We first describe the shear beam theory. This beam thoery is usually called Timoshenko beam theory. The major assumption of this theory is that the normal planes are assumed to remain plane, i.e. flat. Thus the planes normal to the midline rotate as rigid bodies. Consider the motion of a point $P$ whose orthogonal projection on the midline is point $C$. If the normal plane rotates as a rigid body, the velocity of point $P$ relative to the velocity of point $C$ is given by

$$
\begin{equation*}
\mathbf{v}_{C P}=\omega \times \mathbf{r} \tag{9.2.1a}
\end{equation*}
$$

where $\omega$ is the angular velocity of the plane and $\mathbf{r}$ is the vector from $C$ to $P$. In two dimensions, the only nonzero component of the angular velocity vector of the plane is the z-component, so $\omega=\dot{\theta} \mathbf{e}_{z} \equiv \omega \mathbf{e}_{z}$. Since $\mathbf{r}=y \mathbf{e}_{y}$, the relative velocity is

$$
\begin{equation*}
\mathbf{v}_{C P}=\omega \times \mathbf{r}=-y \omega \mathbf{e}_{x} . \tag{9.2.1b}
\end{equation*}
$$

The velocity of any point along the midline is only a function of $x$, so

$$
\begin{equation*}
\mathbf{v}^{M}(x)=v_{x}^{M}(x) \mathbf{e}_{x}+v_{y}^{M}(x) \mathbf{e}_{y} \tag{9.2.1c}
\end{equation*}
$$

The velocity of any point in the beam is then given by adding the relative velocity (9.2.1b) to the midline velocity

$$
\begin{equation*}
\mathbf{v}=\mathbf{v}^{M}(x)+\omega \times \mathbf{r}=\mathbf{v}^{M}(x)-y \omega \mathbf{e}_{x} \tag{9.2.1d}
\end{equation*}
$$

The $x$-component of the total velocity is obtained form the above:

$$
\begin{equation*}
v_{x}(x, y)=v_{x}^{M}(x)-y \omega(x) \tag{9.2.2}
\end{equation*}
$$

where $v_{x}^{M}(x)$ is the x -component of the velocity of the midline and $\dot{\theta}(x)$ is the angular velocity of the normal to the midline. The $y$-component of the velocity is equivalent to that of the midline through the depth of the beam, so

$$
\begin{equation*}
v_{y}(x, y)=v_{y}^{M}(x) \tag{9.2.3}
\end{equation*}
$$

Applying the definition of the rate-of-deformation $D_{i j}=\operatorname{sym}\left(v_{i, j}\right)$, see Section 3.3.2, shows that the rate-of-deformation for a Timoshenko beam is given by

$$
\begin{equation*}
D_{x x}=v_{x, x}^{M}-y \omega_{, x}, D_{y y}=0, D_{x y}=\frac{1}{2}\left(v_{y, x}^{M}-\omega\right) \tag{9.2.4a-c}
\end{equation*}
$$

It can be seen that the only nonzero components of the rate-of-deformation are the axial component, $D_{x x}$, and the shear component, $D_{x y}$, the latter is called the transverse shear.

It can be seen immediately from (9.2.2) and (9.2.3) that the dependent variables $v_{i}^{M}(x)$ and $\theta(x)$ need only be $C^{0}$ for the rate-of-deformation to be finite throughout the beam. Thus the standard isoparametric shape functions can be used in the construction of shear beam finite elements. Theories for which the interpolants need only be $C^{0}$ are often called $C^{0}$ structural theories.
9.2.3. Euler-Bernoulli Theory. In the Euler-Bernoulli or engineering beam theories, the kinematic assumption is that the normal remains normal and straight. Therefore the angular velocity of the normal is given by the rate of change of the slope of the midline

$$
\omega=v_{y, x}^{M}
$$

By examining Eq. (9.2.4c) it can be seen that the above is equivalent to requiring the shear rate-of-deformation $D_{x y}$ to vanish, which implies that the angle between the normal and the midline does not change, i.e. the normal remains normal. The axial displacement is then given by

$$
v_{x}(x, y)=v_{x}^{M}(x)-y v_{y, x}^{M}(x)
$$

The rate-of-deformation in Euler-Bernoulli (or engineering) beam theory is given by

$$
D_{x x}=v_{x, x}^{M}-y v_{y, x x}^{M}, \quad D_{y y}=0, D_{x y}=0
$$

Two features are noteworthy in the above:

1. the transverse shear vanishes;
2. the second derivative of the velocity appears in the expression for the rate-ofdeformation tensor, so the velocity field must be $C^{1}$ for the rate-ofdeformation to be well-defined.
Whereas in the Timoshenko beam, two dependent variables are needed, only a single dependent variable is needed for the Euler-Bernoulli beam. Similar reductions in the number of unknowns take place in the corresponding shell theories: a Kirchhoff-Love shell theory only has three dependent variables, whereas a Mindlin-Reissner theory has five dependent variables (six are often used in practice; this is discussed in Section 9.4. This type of structural theory is often called a $C^{1}$ theory because of the need for $C^{1}$ approximations. The requirement for $C^{1}$ approximations is the biggest disadvantage of Euler-Bernoulli and Kirchhoff-Love theories, since $C^{1}$ approximations are difficult to construct in more than one dimension. For this reason, $C^{1}$ structural theories are seldom used in software except for beams. Beam elements are often based on Euler-Bernoulli theory because $C^{1}$ interpolants are easily constructed in one dimension. Theories which require $C^{1}$ interpolants are often called $C^{1}$ structural theories.

Transverse shear is of significance only in thick beams. However Timoshenko beams Mindlin-Reissner shells are frequently used even when transverse shear is not physically important. For thin beams, the transverse shears in Timoshenko beams also go to zero in well-behaved elements. Thuis the normality hypothesis, which implies that transverse shear vanishes for thim beams, is a trend also observed in numerical solutions and analytic solutions as the thickness decreases.
9.2.4. Discrete Kirchhoff and Mindlin-Reissner Theories. A third approach, which is only used in numerical method, are the discrete theories. In the discrete Kirchhoff theory, the Kirchhoff-Love assumption is only applied discretely, i.e. at a finite number of points, usually the quadrature points. Transverse shear then develops at other points in the element but it is ignored. Similarly, discrete MindlinReissner elements can be formulated by imposing these assumptions discretely.

### 9.3 DEGENERATED CONTINUUM BEAM.

In the following, the continuum based (CB) formulation for a beam in two dimensions is developed. In this development we will impose the kinematic assumptions on the discrete equations, i.e. the continuum finite element will be modified so that it behaves like a shell. In the next Section, we will develop the CB beam by imposing the kinematic assumption on the motion before writing the weak form. These two sections will introduce many of the concepts and techniques which are used in the development of CB shell elements. The elements to be developed are applicable to nonlinear materials and geometrical nonlinearities. Either an updated Lagrangian or a total Lagrangian approach can be used. However, Lagrangian elements are almost always used for shells and structures because they consist of closely separated surfaces which are difficult to treat with Eulerian elements.

We will not go through the steps followed in Chapters 2, 4, and 7 of developing a weak form for the momentum equation and showing the equivalence to the strong form, since we will use the discrete equations for continua. The essence of the CB beam
approach is to impose the kinematic assumption on the motion of continuum elements. We will first describe how this is done directly on the discrete continuum equations.


Figure 9.3. A three-node CB beam element and the underlying 6-node continuum element; the two notations for slave nodes of the underlying continuum element by two conventions are shown with the initial and current configurations.
9.3.1. Definitions and Nomenclature. A finite element model of a CB beam is shown in Figure 9.3; a 6-node quadrilateral is shown here as the underlying continuum element, but any other contiuum element with $n_{N}$ nodes on the top and bottom surfaces can also be used. The parent element for the continuum element is also shown. As can be seen in Fig. 9.3, the continuum element only has nodes on the top and bottom surfaces (the surfaces are lines in two dimensional elements), for as will become clear, the motion must be linear in $\eta$. The reference line may be placed anywhere, but we will place it on the line $\eta=0$ for convenience.

The lines of constant $\xi$ are called fibers (they are also called pseudonormals), the unit vector along each fiber is called a director, which is denoted by $\mathbf{p}$. The directors play the same role in the CB theory as normals in the classical Mindlin-Reissner theory, hence the alternate name pseudonormals. Lines of constant $\eta$ are called lamina.

Master nodes are introduced at the intersections of the fibers connecting nodes of the continuum element with the reference line. The degrees-of-freedom of these nodes describe the motion of the beam, and the equations of motion will be formulated in terms of generalized forces and velocities at these nodes. The original nodes of the continuum element on the top and bottom surfaces are designated as slave nodes. Each master node is associated with a pair of slave nodes along a common fiber, see Fig. 9.3. The slave
nodes are indicated either by superposed bars or by superscript plus and minus signs on the node numbers: thus node $I^{+}$and $I^{-}$are slave nodes associated with master node $I$ and lie on the top $(+)$ and bottom (-) surfaces of the beam; $I^{*}$ are alternate node numbers of the continuum element. Each triplet of nodes $I^{-}, I$, and $I^{+}$is collinear and lie on the same fiber. The appellations "top" and "bottom" have no exact definition; either surface of the beam can be designated as the "top" surface.

The two sets of node numbers for the continuum element are related by.

$$
\begin{array}{ll}
I^{*}=I^{+} & I^{+}=I^{*} \text { for } I^{*} \leq n_{N}  \tag{9.3.0}\\
I^{*}=I^{-}+n_{N} & I^{-}=I^{*}-n_{N} \text { for } I^{*}>n_{N}
\end{array}
$$

For each point in the beam, a corotational coordinate system is defined with $x$ tangent to the lamina; $y$ then corresponds to the normal direction.
9.3.2. Assumptions. The following assumptions are made:

1. the fibers remain straight;
2. the element is in a state plane stress, so

$$
\begin{equation*}
\hat{\sigma}_{y y}=0 \tag{9.3.1}
\end{equation*}
$$

3. the elongation of fibers is governed by conservation of matter and/or the constitutive eqaution
The first assumption will be called the modified Mindlin-Reissner assumption in this book. It differs from what we call the classical Mindlin-Reissner assumption, which requires the normal to remain straight; the fibers are not initially normal to the midline. The resulting theory is similar to a single director Cosserat theory. Although the shear beam theory is called a Timoshenko beam theory, we will use the appellation modified Mindlin-Reissner for this assumptions for both beams and shells.

For the CB beam element to satisfy the classical Mindlin-Reissner assumptions, it is necessary for the fibers be aligned as closely as possible with the normal to the midline. This can be accomplished by placing the slave nodes so that the fibers are as close to normal to the midline as possible in the initial configuration. Otherwise the behavior of the degenerated beam element may deviate substantially from classical Mindlin-Reissner theory and may not agree with the physical behavior of beams. From exercise, it can be seen that it is impossible to align the fibers with the normal exactly along the entire length of the element when the motion of the continuum element is $C^{0}$.

Instead of the third assumption, many authors assume that the fibers are insxtensible. Inextensibility contradicts the plane stress assumption: the fibers are usually close to the $\hat{y}$ direction and so if $\hat{\sigma}_{y y}=0$, the velocity strain in the $\hat{y}$ direction generally can not vanish. The contradiction is reconciled by not using the continuum displacement field to compute $\hat{D}_{y y}$; instead, $\hat{D}_{y y}$ is computed by the constitutive eqaution from the requirement that $\hat{\sigma}_{y y}=0$.

The assumption of constant fiber length is inconsistent with the conservation of matter: if the beam element is stretched, it must become thinner to conserve matter. Conservation of matter is usually imposed through the constitutive equation. For
example, in plasticity, conservation of matter is reflected in the isochoric character of the plastic strains, see Chapter 5. Therefore, if the thickness strain is calculated through the constitutive equation via the plane stress requirement, conservation of matter is enforced. The important feature of the third assumption is that the extension of the fibers is not governed by the equations of motion or equilibrium. From the third assumption, it follows automatically that the equations of motion or equilibrium associated with the thickness modes are eliminated from the system.

The third assumption can be replaced by an inextensibility assumption if the change is thickness is small. In that case, the thickness velocity strain $\hat{D}_{y y}$ is still computed by the constitutive equation, but the effect of the thickness strain on the position of the slave nodes is neglected, so that the nodal internal forces do not reflect changes in the thickness. The theory is then applicable only to problems with moderate strains (on the oder of 0.01). This approach is taken in the following description of beam motion. In Section 9.5 we describe a methodlogy that completely accounts for thickness strains.

We have not given the plane stress condition in terms of the PK2 stress or nominal stress, for unless simplifying assumptions are made, they are more complex than (9.3.1): the plane stress condition requires that the $\hat{y}$-component of the physical stress vanish, which is not equivalent to requiring $\hat{S}_{22}$ to vanish. However, since the plane stress requirement is only an assumption which is almost never satisfied exactly in physical beams, the use of the slightly different condition $\hat{S}_{22}=0$ is often acceptable, particularly for thin beams where $\mathbf{p}$ and $\hat{y}$ are collinear. This is examined further in Exercise 9.?.
9.4.3. Motion. The motion of the beam is described by translations of the master nodes, $x_{I}(t), y_{I}(t)$ and rotations of the nodal fibers, which are denoted by $\theta_{I}(t)$. To develop this form of the motion, we begin with the motion of the element in terms of the slave node (the nodes of the underlying continuum element) position vectors by

$$
\begin{equation*}
\mathbf{x}(\xi, t)=\sum_{I^{+}=1}^{n_{N}} \mathbf{x}_{I^{+}}(t) N_{I^{+}}(\xi, \eta)+\sum_{I^{-}=1}^{n_{N}} \mathbf{x}_{I^{-}}(t) N_{I^{-}}(\xi, \eta)=\sum_{I^{+}=1}^{2 n_{N}} \mathbf{x}_{I^{*}}(t) N_{I^{*}}(\xi, \eta) \tag{9.3.2}
\end{equation*}
$$

In the above $\mathbf{x}^{T}=[x, y], N_{I^{*}}(\xi, \eta)$ are the standard shape functions for continua (indicated by asterisks or superscripts " + " and "-" signs on nodal index) and $n_{N}$ is the number of nodes along the top or bottom surface.

The shape functions of the underlying continuum must be linear in $\eta$ for the above motion to be consistent with the modified Mindlin-Reissner assumption. Therefore the parent element can only have two nodes along the $\eta$ direction, i.e. there can be only two slave nodes along a fiber. The velocity field is obtained by taking the material time derivative of the above, which gives

$$
\begin{equation*}
\mathbf{v}(\xi, t)=\sum_{I^{+}=1}^{n_{N}} \mathbf{v}_{I^{+}}(t) N_{I^{+}}(\xi, \eta)+\sum_{I^{-}=1}^{n_{N}} \mathbf{v}_{I^{-}}(t) N_{I^{-}}(\xi, \eta)=\sum_{I=1}^{2 n_{N}} \mathbf{v}_{I^{*}}(t) N_{I^{*}}(\xi, \eta) \tag{9.3.2b}
\end{equation*}
$$

We now impose the inextensibility assumption and the modified Mindlin-Reissner assumptions on the motion of the slave nodes

$$
\begin{equation*}
\mathbf{x}_{I^{+}}(t)=\mathbf{x}_{I}(t)+\frac{1}{2} h_{I}^{0} \mathbf{p}_{I}(t) \quad \mathbf{x}_{I^{-}}(t)=\mathbf{x}_{I}(t)-\frac{1}{2} h_{I}^{0} \mathbf{p}_{I}(t) \tag{9.3.3}
\end{equation*}
$$

where $\mathbf{p}_{I}(t)$ is the director at master node $I$, and $h_{I}^{0}$ is the initial thickness of the beam at node $I$ (or more precisely a pseudo-thickness since it is the distance between the top to bottom surfaces along a fiber, not along the normal). The director at node $I$ is a unit vector along the fiber $\left(I^{-}, I, I^{+}\right)$, so the current nodal directors are given by

$$
\begin{equation*}
\mathbf{p}_{I}(t)=\frac{1}{h_{I}^{0}}\left(\mathbf{x}_{I^{+}}(t)-\mathbf{x}_{I^{-}}(t)\right)=\mathbf{e}_{x} \cos \theta_{I}+\mathbf{e}_{y} \sin \theta_{I} \tag{9.3.4a}
\end{equation*}
$$

where $\mathbf{e}_{x}$ and $\mathbf{e}_{y}$ are the global base vectors. The above can also be derived by subtracting (9.3.3b) from (9.3.3a). The initial nodal directors are

$$
\mathbf{p}_{I}^{0}(t)=\frac{1}{h_{I}^{0}}\left(\mathbf{X}_{I^{+}}-\mathbf{X}_{I^{-}}\right)=\mathbf{e}_{x} \cos \theta_{I}^{0}+\mathbf{e}_{y} \sin \theta_{I}^{0}
$$

The initial thickness is given by

$$
\begin{equation*}
h_{I}^{0}=\left\|\mathbf{x}_{I+}(0)-\mathbf{x}_{I-}(0)\right\| \tag{9.3.4c}
\end{equation*}
$$

From. (9.3.3) it can be shown that if $h_{I}=h_{I}^{0}$, then the fiber through node $I$ is inextensible, i.e. $\left\|x_{I^{+}}-x_{\Gamma}\right\|$ is constant during the motion; it will be shown in Section 9.4 that all fibers of the element remain constant in length when the nodal fibers remain constant in length.

The velocities of the slave nodes are obtained by taking the material time derivative of (9.3.3), yielding

$$
\begin{equation*}
\mathbf{v}_{I+}(t)=\mathbf{v}_{I}(t)+\frac{1}{2} h_{I}^{0} \omega_{I}(t) \times \mathbf{p}_{I}(t) \quad \mathbf{v}_{I-}(t)=\mathbf{v}_{I}(t)-\frac{1}{2} h_{I}^{0} \omega_{I}(t) \times \mathbf{p}_{I}(t) \tag{9.3.5}
\end{equation*}
$$

where we have used (9.2.1) to express the nodal velocities in terms of the angular velocities, noting that the vectors from the master node to the slave nodes are $\frac{1}{2} h_{I}^{0} \mathbf{p}_{I}(t)$ and $-\frac{1}{2} h_{I}^{0} \mathbf{p}_{I}(t)$ for the top and bottom slave nodes, respectively. Since the model is twodimensional, $\omega=\omega_{z} \mathbf{e}_{z} \equiv \dot{\theta} \mathbf{e}_{z}$ and the slave node velocity can be written by using (9.3.4a), (9.3.4b), and (9.3.5) as:

$$
\begin{align*}
& \mathbf{v}_{I^{+}}=\mathbf{v}_{I}-\omega_{z}\left(\left(y_{I^{+}}-y_{I}\right) \mathbf{e}_{x}-\left(x_{I^{+}}-x_{I}\right) \mathbf{e}_{y}\right)=\mathbf{v}_{I}-\frac{1}{2} \omega_{z} h_{I}^{0}\left(\mathbf{e}_{x} \sin \theta-\mathbf{e}_{y} \cos \theta\right)  \tag{9.3.6a}\\
& \mathbf{v}_{I^{-}}=\mathbf{v}_{I}-\omega_{z}\left(\left(y_{I^{-}}-y_{I}\right) \mathbf{e}_{x}-\left(x_{I^{-}}-x_{I}\right) \mathbf{e}_{y}\right)=\mathbf{v}_{I}-\frac{1}{2} \omega_{z} h_{I}^{0}\left(\mathbf{e}_{x} \sin \theta-\mathbf{e}_{y} \cos \theta\right) \tag{9.3.6b}
\end{align*}
$$

The motion of the master nodes is described by three degrees of freedom per node

$$
\mathbf{d}_{I}(t)=\left[\begin{array}{lll}
u_{x I}^{M} & u_{y I}^{M} & \theta_{I}
\end{array}\right]^{T} \quad \dot{\mathbf{d}}_{I}(t)=\left[\begin{array}{lll}
v_{x I}^{M} & v_{y I}^{M} & \omega_{I} \tag{9.3.6}
\end{array}\right]^{T}
$$

Equation (9.3.6) can be written in matrix form as

$$
\left\{\begin{array}{l}
\mathbf{v}_{I^{+}}  \tag{9.3.7a}\\
\mathbf{v}_{I^{-}}
\end{array}\right\}^{\text {slave }}=\left\{\begin{array}{l}
v_{x I^{+}} \\
v_{y I^{+}} \\
v_{x I^{-}} \\
v_{y I^{-}}
\end{array}\right\}=\mathbf{T}_{I} \dot{\mathbf{d}}_{I}
$$

Recall that we are not using the summation convention on nodal indices in this Chapter. From a comparison of (9.3.7a) and (9.3.6) we can see that

$$
\mathbf{T}_{I}=\left[\begin{array}{ccc}
1 & 0 & y_{I}-y_{I^{+}}  \tag{9.3.7b}\\
0 & 1 & x_{I^{+}}-x_{I} \\
1 & 0 & y_{I}-y_{I^{-}} \\
0 & 1 & x_{I^{-}}-x_{I}
\end{array}\right\} \quad \dot{d}_{I}=\left\{\begin{array}{c}
v_{x I} \\
v_{y I} \\
\omega_{I}
\end{array}\right\}
$$

The velocities of the master nodes are the degrees of freedom of the discrete model. We can see from the above that the discrete variables characterizing the motion of the beam are the two components of the velocity of the midline and the angular velocity of the node.
9.2.4.3. Nodal Forces. The procedure for calculating the internal nodal forces at the slave nodes in the CB approach is almost identical to that of the continuum element. The nodal velocities of the underlying continuum element are obtained from the master nodal velocities by (9.3.7). The continuum element procedures as described in Chapter 4 are then used to obtain the nodal internal forces at the slave nodes via the strain-displacement and constitutive equations.

The master nodal internal forces are related to the slave nodal internal forces by the transformation rule given in Section 4.5.6, Eq. (4.5.36). Since the slave nodal velocities are related to the master nodal velocities by (9.3.7), the nodal forces are related by

$$
\mathbf{f}_{I}^{\text {mast }}=\left\{\begin{array}{l}
f_{x I}  \tag{9.3.8}\\
f_{y I} \\
m_{I}
\end{array}\right\}=\mathbf{T}_{I}^{T}\left\{\begin{array}{l}
\mathbf{f}_{I+} \\
\mathbf{f}_{I-}
\end{array}\right\}^{\text {slave }}=\left[\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
y_{I}-y_{I^{+}} & x_{I^{+}}-x_{I} & y_{I}-y_{I^{-}} & x_{I^{-}}-x_{I}
\end{array}\right]\left\{\begin{array}{l}
f_{x I+} \\
f_{y I+} \\
f_{x I-} \\
f_{y I-}
\end{array}\right\}
$$

The external nodal forces at the master nodes can be obtained from the slave node external forces by the same transformation. The column matrix of nodal forces consists of the two force components $f_{x I}$ and $f_{y I}$ and the moment $m_{I}$. It can readily be seen that
they are conjugate in power to the velocities of the master nodes, i.e. the power of the forces at node $I$ is given by $\mathbf{v}_{I} \cdot \mathbf{f}_{I}$; the superscripts "mast" have been dropped.

The major difference from the procedures in the standard continuum element is that in the evaluation of the constitutive law for the CB beam, the plane stress assumption (9.3.1) must be observed. Therefore, it is convenient to transform components of the stress and velocity strain tensors at each point of the beam to the corotational coordinate systems $\hat{x}, \hat{y}$,. For this purpose, local base vectors $\hat{\mathbf{e}}_{i}$ are constructed so that $\hat{\mathbf{e}}_{x}$ is tangent to the lamina and $\hat{\mathbf{e}}_{y}$ is normal to the lamina, see Fig. 9.4.


Figure 9.4 Schematic of DC beam showing lamina, the corotational unit vectors $\hat{\mathbf{e}}_{x}, \hat{\mathbf{e}}_{y}$ and the director $\mathbf{p}(\xi, t)$ at the ends; note $\mathbf{p}$ usually does not coincide with $\hat{\mathbf{e}}_{y}$.

The base vectors at any point are given by

$$
\begin{array}{ll}
\hat{\mathbf{e}}_{x}=\frac{x_{\xi \xi} \mathbf{e}_{x}+y_{\prime_{\xi}} \mathbf{e}_{y}}{\left(x_{\xi \xi}^{2}+y_{, \xi}^{2}\right)^{1 / 2},} & \hat{\mathbf{e}}_{y}=\frac{-y_{\xi} \mathbf{e}_{x}+x_{\xi} \mathbf{e}_{y}}{\left(x_{\xi}^{2}+y_{I_{\xi}}^{2}\right)^{1 / 2}}  \tag{9.3.9}\\
x, \xi=\sum_{I} x_{I^{*}} N_{I^{*}, \xi}(\xi, \eta) & y_{, \xi}=\sum_{I} y_{I^{*}} N_{I^{*}, \xi}(\xi, \eta)
\end{array}
$$

The rate-of-deformation is transformed to the corotational system by Box 3.2?????:

$$
\hat{\mathbf{D}}=\mathbf{R}^{T} \mathbf{D R} \quad \text { where } \mathbf{R}=\left[\begin{array}{ll}
\mathbf{e}_{x} \cdot \hat{\mathbf{e}}_{x} & \mathbf{e}_{x} \cdot \hat{\mathbf{e}}_{y}  \tag{9.3.10}\\
\mathbf{e}_{y} \cdot \hat{\mathbf{e}}_{x} & \mathbf{e}_{y} \cdot \hat{\mathbf{e}}_{y}
\end{array}\right]
$$

In the evaluation of the stress, the plane stress constraint $\hat{\sigma}_{y y}=0$ must be observed. If the constitutive equation is in rate form, the constraint is expressed in the rate form $D \hat{\sigma}_{y y} / D t=0$. For example, for an isotropic hyperelastic material, the stress rate is given by LIU, CORRECTION NEEDS TO BE PUT IN

$$
\frac{D}{D t}\{\hat{\sigma}\}=\frac{D}{D t}\left\{\begin{array}{c}
\hat{\sigma}_{x x}  \tag{9.3.11}\\
\hat{\sigma}_{y y} \\
\hat{\sigma}_{x y}
\end{array}\right\}=\frac{D}{D t}\left\{\begin{array}{c}
\hat{\sigma}_{x x} \\
0 \\
\hat{\sigma}_{x y}
\end{array}\right\}=\frac{E^{\sigma \mathcal{G}}}{1-v^{2}}\left[\begin{array}{ccc}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & \frac{1}{2}(1-v)
\end{array}\right]\left\{\begin{array}{c}
\hat{D}_{x x} \\
\hat{D}_{y y} \\
2 \hat{D}_{x y}
\end{array}\right\}
$$

In the above, the rate form of the plane stress condition $D \hat{\sigma}_{y y} / D t=0$ has been imposed to give $\hat{D}_{y y}=-\mathrm{v} \hat{D}_{x x}$. Solving for the two other components gives

$$
\begin{equation*}
\frac{D \hat{\sigma}_{x x}}{D t}=E^{\sigma \mathcal{G}} \hat{D}_{x x}, \quad \frac{D \hat{\sigma}_{x y}}{D t}=\frac{E^{\sigma \mathcal{G}}}{2(1+v)} \hat{D}_{x y} \tag{9.3.12}
\end{equation*}
$$

As seen in the above, in an isotropic material, the rate of the axial stress is related to the axial rate-of-deformation by the tangent modulus $E^{\sigma \mathcal{G}}$ for the Green-Naghdi rate.

For more general materials (including laws which lack symmetry in the moduli, such as nonassociated plasticity) the rate relation for the stress can be expressed as

$$
\frac{D}{D t}\left\{\begin{array}{c}
\hat{\sigma}_{x x}  \tag{9.3.13}\\
0 \\
\hat{\sigma}_{x y}
\end{array}\right\}=\left[\begin{array}{lll}
\hat{C}_{11} & \hat{C}_{12} & \hat{C}_{13} \\
\hat{C}_{21} & \hat{C}_{22} & \hat{C}_{23} \\
\hat{C}_{31} & \hat{C}_{32} & \hat{C}_{33}
\end{array}\right]^{\sigma}\left\{\begin{array}{c}
\hat{D}_{x x} \\
\hat{D}_{y y} \\
2 \hat{D}_{x y}
\end{array}\right\}
$$

where $\hat{\mathbf{C}}$ is matrix of instantaneous moduli for the Green-Naghdi rate of Cauchy stress, as in plastic models given in Chapter 5, and the second equation enforces the plane stress condition.

The stress $\hat{\sigma}$ can be considered corotational, since the base vectors $\left(\hat{\mathbf{e}}_{x}, \hat{\mathbf{e}}_{y}\right)$ rotate almost exactly with the material. The rotation given through (9.3.9) differs somewhat from that given by a polar decomposition, but is usually a better rotation for composite or reinforced beams than that given by polar decomposition. The fibers of a composite and reinforcements tend to remain aligned with the lamina, and with this rotation, the tangent modulus $\hat{C}_{11}$ pertains to the lamina direction. If the system ( $\hat{\mathbf{e}}_{x}, \hat{\mathbf{e}}_{y}$ ) is not a good enough approximation of the rotation, it can be set by the polar decomposition theorem.


Figure 9.5. A stack of quadrature points and examples of axial stress distributions for an elastic-plastic material.

The slave internal nodal forces are obtained by the mechanics of the continuum element Example 4. and the integrals in (E4.2.11) are evaluated by numerical quadrature over the element domain. Neither full quadrature (4.5.27) nor the selective-reduced quadrature given (4.3.34b) can be used in a CB beam. Both quadrature schemes result in shear locking, to be described in Section 9.5. Shear locking can be avoided in this element by using a single stack of quadrature points along the axis $\xi=0$ as shown in Fig. 9.5. The number of quadrature points required in the $\eta$ direction depends on the material law and the accuracy desired. For a nonlinear hyperelastic material law, 3 Gauss quadrature points are often adequate. For an elastic-plastic law, a minimum of 5 quadrature points is needed. Gauss quadrature is not optimal for elastic-plastic laws since the lack of smoothness in the elastic-plastic constitutive response results in stress distributions with discontinuous derivatives, such as shown in Fig. 9.5. Therefore, the trapezoidal rule is often used.

To illustrate the selective-reduced integration procedure which circumvents shear locking, we consider a two-node beam element based on a 4-node quadrilateral continuum element. The nodal forces are obtained by integration with a single stack of quadrature points at $\xi=0$ to avoid shear locking. The nodal forces at the slave nodes are obtained by (see Section 4.5.4):

$$
\left[f_{x I^{*}}, f_{y I^{*}}\right]^{i n t}=\left.\sum_{Q=1}^{n_{Q}}\left(\left[N_{I^{*}, x} N_{I^{*}, y}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y}  \tag{9.3.15}\\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] \bar{w}_{Q} a J_{\xi}\right)\right|_{\left(0, \eta_{Q}\right)}
$$

where $\eta_{Q}$ are the $n_{Q}$ quadrature points through the thickness of the beam, $\bar{w}_{Q}$ are the quadrature weights, $a$ is the dimension of the beam in the z-direction and $J_{\xi}$ is the Jacobian determinant with respect to the parent element coordinates, (4.4.38). Note that the node numbers $I^{*}$ can be related to the triplet number by Eq. (9.3.0) so the relationship to Eq. (9.3.8) is easily established. The stresses must be rotated back to the global system prior to evaluating the nodal internal forces by (9.3.15). The nodal internal forces can also be evaluated in terms of the corotational system by

$$
\left[f_{\hat{x} I^{*}}, f_{\hat{y} I^{*}}\right]^{\text {int }}=\sum_{Q=1}^{n_{Q}}\left(\left[\begin{array}{ll}
N_{\tilde{I}, \hat{x}} & N_{\bar{I}, \hat{y}}
\end{array}\right]\left[\begin{array}{cc}
\hat{\sigma}_{x x} & \hat{\sigma}_{x y}  \tag{9.3.16}\\
\hat{\sigma}_{x y} & 0
\end{array}\right]\left[\begin{array}{ll}
R_{x x} & R_{y x} \\
R_{x y} & R_{y y}
\end{array}\right] \bar{w}_{Q} a J_{\xi}\right)_{\left(0, \eta_{Q}\right)}
$$

The stress component $\hat{\sigma}_{y y}$ vanishes in (9.316) because of the plane stress condition. The corotational approach is of advantage because the plane stress condition is more easily expressed in corotational components. While the use of the corotational form of the internal forces (9.3.16) eliminates the need to transform the stress components back to the global system after the constitutive update, some of the computational advantage is lost because the shape function derivatives must be evaluated in each corotational system. This computational effort can be reduced by using only one or two corotational systems per stack of quadrature points.

BOX?????
In summary, the procedure for computing the nodal forces in a CB beam element in a corotational, updated Lagrangian approach is:

1. the positions and velocities of the slave nodes are computed by (9.3.3) and (9.3.7) from the positions and velocities of the master nodes;
2. the rate-of-deformation is transformed to the corotational coordinate system at each quadrature point
3. the Cauchy stresses are computed at all quadrature points in the corotational coordinates with the plane stress condition $\hat{\sigma}_{y y}=0$ enforced;
4. the stresses are transformed back to the global coordinates;
5. the nodal internal forces are computed at the slave nodes by standard method for continua, (E.4.2.11) as illustrated by (9.3.15-16);
6. the slave nodal forces are transformed to the master nodes by (9.3.8).
9.2.4.4. Mass Matrix. The mass matrix of the CB beam element can be obtained by using the transformation (4.5.39) using for $\hat{\mathbf{M}}$ the mass matrix for the underlying continuum element.

$$
\begin{equation*}
\mathbf{M}=\mathbf{T}^{T} \hat{\mathbf{M}} \mathbf{T} \tag{9.3.18a}
\end{equation*}
$$

where

$$
\mathbf{T}=\left[\begin{array}{cccc}
\mathbf{T}_{1} & \mathbf{0} & . & \mathbf{0}  \tag{9.3.18b}\\
\mathbf{0} & \mathbf{T}_{2} & \cdot & \mathbf{0} \\
\cdot & \cdot & \cdot & \cdot \\
\mathbf{0} & \mathbf{0} & . & \mathbf{T}_{n_{N}}
\end{array}\right]
$$

This mass matrix does not account for the time dependence of the $\mathbf{T}$ matrix. If we account for the time dependence of $\mathbf{T}$, the inertial force according to (4.5.42) is given by

$$
\begin{equation*}
\mathbf{f}^{\text {inert }}=\mathbf{T}^{T} \hat{\mathbf{M}} \dot{\mathbf{v}}+\mathbf{T}^{T} \hat{\mathbf{M}} \dot{\mathbf{T}} \mathbf{v} \tag{9.3.17}
\end{equation*}
$$

and $\hat{\mathbf{M}}$ is given in Example 4.2 and $\mathbf{T}_{I}$ is given by (9.3.7). The matrix $\dot{\mathbf{T}}_{I}$ is obtained by taking a time derivative of (9.3.7b) and using the fact that for node $I$, $\frac{d}{d t}\left(\omega_{I} \times \mathbf{r}_{I}\right)=\omega_{I} \times\left(\omega_{I} \times \mathbf{r}_{I}\right)$, which gives

$$
\dot{\mathbf{T}}_{I}=\omega\left[\begin{array}{ccc}
1 & 0 & x_{I}-x_{I^{+}}  \tag{9.3.19}\\
0 & 1 & y_{I}-y_{I^{+}} \\
1 & 0 & y_{I}-y_{I^{=}} \\
0 & 1 & x_{I}-x_{I^{=}}
\end{array}\right]
$$

From (9.3.17) and (9.3.19), it can be seen that the acceleration of the CB element will include a term proportional to the square of the angular velocity. Consequently the inertial term in the discrete ordinary differential equations are no longer linear in the velocities and the time integration of the equations becomes more complex. This second term in (9.3.17) is usually neglected.

Either the consistent or lumped mass of the continuum element, $\hat{\mathbf{M}}$, can be used to generate the mass matrix for the CB beam element. Equation (9.3.18a) does not yield a diagonal matrix even when the diagonal mass matrix of the continuum element is used.

Two techniques are used to obtain diagonal matrices:

1. The consistent mass matrix of the quadrilateral is transformed by (4.5.39) and the row sum technique is used.
2. The translational masses of the diagonal mass matrix are taken to be half the mass of the element and the rotational mass is taken to be the rotational inertia of half the beam about the node.

For a CB beam based on a rectangular 4-node continuum element, the second procedure yields (this is left as an exercize)

$$
\mathbf{M}=\frac{\rho h_{I}^{0} \ell_{0} a_{0}}{420}\left[\begin{array}{cccccc}
210 & 0 & 0 & 0 & 0 & 0  \tag{9.3.20}\\
0 & 210 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha \ell_{0}^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 210 & 0 & 0 \\
0 & 0 & 0 & 0 & 210 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha \ell_{0}^{2}
\end{array}\right]
$$

where $\alpha$ is often treated as a scale factor for the rotational inertia. This scale factor is chosen in explicit codes so that the stable time step depends only on the translational degrees of freedom, see Key and Beisinger (1971). LIU FILL IN
9.2.?. Equations of Motion. The equations of motion at a node are given by

$$
\begin{equation*}
\mathbf{M}_{I J} \dot{\mathbf{v}}_{J}+\mathbf{f}_{I}^{\text {int }}=\mathbf{f}_{I}^{\text {ext }} \text { sum on } J \tag{9.3.21}
\end{equation*}
$$

where the nodalo forces and nodal displacements

$$
\mathbf{f}_{I}=\left\{\begin{array}{c}
f_{x I}  \tag{9.3.22}\\
f_{y I} \\
m_{I}
\end{array}\right\} \quad \dot{\mathbf{d}}_{I}=\left\{\begin{array}{c}
v_{x I} \\
v_{y I} \\
\omega_{I}
\end{array}\right\}
$$

which are the master degrees of freedom, i.e. The equations are identical in form to (4.??). For a diagonal mass matrix the equations can be when written out as

$$
\left[\begin{array}{ccc}
M_{11} & 0 & 0  \tag{9.3.23}\\
0 & M_{22} & 0 \\
0 & 0 & M_{33}
\end{array}\right]_{I I}\left\{\begin{array}{l}
\dot{v}_{x I} \\
\dot{v}_{y I} \\
\dot{\omega}_{I}
\end{array}\right\}^{2}+\left\{\begin{array}{l}
f_{x I} \\
f_{y I} \\
m_{I}
\end{array}\right\}^{\text {ext }}=\left\{\begin{array}{l}
f_{x I} \\
f_{y I} \\
m_{I}
\end{array}\right\}^{\text {int }}
$$

where $M_{i i}, i=1$ to 3 are the assembled diagonal masses at node $I$. Although we have not derived these equations explicitly, they follow from (4.??) since we have only made transformation of variables. Showing this is left as an exercise. For equilibrium processes, the first term is dropped.

Tangent Stiffness. The tangential and load stiffnesses are obtained from the corresponding matrices for the underlying continuum element by the transformation (4.5.43). However, the continuum stiffnesses must reflect the plane stress assumption. This is illustrated in Example 9.1. These matrices do not need to be rederived for CB beams.

### 9.4. ANALYSIS OF CB BEAM

In order to obtain a better understanding of the CB beam, it is worthwhile to examine its motion from a viewpoint which more closely parallels classical beam theory. The analysis in this Section leads to discrete equations which are identical to those described in the previous section. It is more pleasing conceptually, but working in this framework is more burdensome, since the many of the entities needed for a standard implementation, such as the tangent stiffness and the mass matrix, have to be developed from scratch, whereas in the previous approach they are inherited from a continuum element with small modifications.

We start with the description of the motion. Recall that in the underlying continuum element, there are only two slave nodes along any fiber, i.e. in the thickness direction of the beam, so that the motion is linear in $\eta$. Consequently we can describe the motion of the CB beam by

$$
\begin{equation*}
\mathbf{x}(\xi, \eta, t)=\mathbf{x}^{M}(\xi, t)+\bar{\eta}(\xi, \eta) \mathbf{p}(\xi, t) \tag{9.4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\eta}(\xi, \eta)=\frac{1}{2} \eta h^{0}(\xi) \tag{9.4.2}
\end{equation*}
$$

The independent variables $\xi$ and $\eta$ are curvilinear coordinates with $\eta=0$ corresponding to the reference line. The top and bottom surfaces of the beam are given by $\eta=1$ and
$\eta=-1$, respectively. Note that although we use the same nomenclature for the curvilinear coordinates as for the parent element coordinates, (9.4.1) is independent of a parent element and $\xi$ and $\eta$ are an arbitrary set of curvilinear coordinates. The initial configuration is given by writing (9.4.1) at the initial time:

$$
\begin{equation*}
\mathbf{X}(\xi, \eta)=\mathbf{X}^{M}(\xi)+\bar{\eta}(\xi, \eta) \mathbf{p}_{0}(\xi) \tag{9.4.3}
\end{equation*}
$$

where $\mathbf{p}_{0}(\xi)$ is the initial director and $\mathbf{X}^{M}(\xi)$ describes the initial reference line.
In this form of the motion, it is straightforward to show that all fibers are inextensible if the nodal fibers are inextensible. The length of a fiber is given by the distance between the top and bottom surfaces along the fiber, i.e. the distance between the points at $\eta=-1$ and $\eta=1$ for a constant value of $\xi$. Using (9.4.3) it follows that the length of any fiber in the deformed configuration is given by

$$
\begin{aligned}
\|\mathbf{x}(\xi, 1, t)-\mathbf{x}(\xi,-1, t)\| & =\left\|\left(\mathbf{x}^{M}(\xi, t)+\frac{h^{0}(\xi)}{2} \mathbf{p}(\xi, t)\right)-\left(\mathbf{x}^{M}(\xi, t)-\frac{h^{0}(\xi)}{2} \mathbf{p}(\xi, t)\right)\right\| \\
& =\left\|h^{0}(\xi) \mathbf{p}(\xi, t)\right\|=h^{0}(\xi)
\end{aligned}
$$

where the last step follows from the fact that the director $\mathbf{p}$ is a unit vector. Hence the length of a fiber is always $h^{0}(\xi)$.

The displacement is obtained by subtracting (9.4.3) from (9.4.1), which gives

$$
\begin{equation*}
\mathbf{u}(\xi, \eta, t)=\mathbf{u}^{M}(\xi, t)+\bar{\eta}(\xi, \eta)\left(\mathbf{p}(\xi, t)-\mathbf{p}_{0}(\xi)\right) \tag{9.4.4}
\end{equation*}
$$

Because the directors are unit vectors, the second term on the RHS of the above is a function of a single variable, the angle $\theta(\xi, t)$, which is measured counterclockwise from the $x$-axis as shown in Fig. 9.4. This can be clarified by expressing the second term of (9.4.4) in terms of the global base vectors:

$$
\begin{equation*}
\mathbf{u}=\mathbf{u}^{M}+\bar{\eta}\left(\mathbf{e}_{x}\left(\cos \theta-\cos \theta_{0}\right)+\mathbf{e}_{y}\left(\sin \theta-\sin \theta_{0}\right)\right) \tag{9.4.5}
\end{equation*}
$$

$\theta_{0}(\xi)$ is the initial angle of the director at $\xi$. The velocity is the material time derivative of the displacement (9.4.5):

$$
\begin{equation*}
\mathbf{v}(\xi, \eta, t)=\mathbf{v}^{M}(\xi, t)+\bar{\eta}(\xi, \eta) \dot{\mathbf{p}}(\xi, t) \tag{9.4.6}
\end{equation*}
$$

Using (9.2.1a), the above can be written

$$
\begin{equation*}
\mathbf{v}=\mathbf{v}^{M}+\bar{\eta} \omega \times \mathbf{p} \tag{9.4.7}
\end{equation*}
$$

where $\omega(\xi, t)$ is the angular velocity of the director. Noting as before that the only nonzero component of this angular velocity is normal to the plane, the vectors are expressed in terms of the base vectors as follows

$$
\begin{equation*}
\omega=\omega \mathbf{e}_{z} \quad \mathbf{p}=\hat{\mathbf{e}}_{x} \cos \hat{\theta}+\hat{\mathbf{e}}_{y} \sin \hat{\theta} \quad \mathbf{v}^{M}=\hat{v}_{x}^{M} \hat{\mathbf{e}}_{x}+\hat{v}_{y}^{M} \hat{\mathbf{e}}_{y} \tag{9.4.7.b}
\end{equation*}
$$

where $\hat{\theta}$ is the angle between the tangent and the director, as shown in Fig. 9.6.


Figure 9.6 Nomenclature for CB beam in two dimensions showing director $\mathbf{p}$ and normal $\mathbf{n}$.
The velocity can then be written as

$$
\begin{equation*}
\mathbf{v}=\hat{v}_{x}^{M} \hat{\mathbf{e}}_{x}+\hat{v}_{y}^{M} \hat{\mathbf{e}}_{y}+\bar{\eta} \omega\left(-\hat{\mathbf{e}}_{x} \sin \hat{\theta}+\hat{\mathbf{e}}_{y} \cos \hat{\theta}\right) \tag{9.4.8}
\end{equation*}
$$

We define vector $\mathbf{q}$ by

$$
\begin{equation*}
\mathbf{q}=\mathbf{e}_{z} \times \mathbf{p}=-\hat{\mathbf{e}}_{x} \sin \hat{\theta}+\hat{\mathbf{e}}_{y} \cos \hat{\theta} \tag{9.4.10}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\mathbf{v}=\mathbf{v}^{M}+\hat{y} \omega \mathbf{q} \tag{9.4.11}
\end{equation*}
$$

Noting (9.4.2) and Fig. 9.6, it can be seen that

$$
\begin{equation*}
\bar{\eta}=\frac{\hat{y}}{\sin \theta}=\frac{\hat{y}}{\cos \bar{\theta}} \tag{9.4.11b}
\end{equation*}
$$

The corotational components of the velocity are then obtained by writing (9.4.6) in the corotational basis with (9.4.11) used to eliminate the $y$ coordinate:

$$
\left\{\begin{array}{c}
\hat{v}_{x}  \tag{9.4.12}\\
\hat{v}_{y}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{v}_{x}^{M} \\
\hat{v}_{y}^{M}
\end{array}\right\}+\omega \hat{y}\left\{\begin{array}{c}
-1 \\
\tan \bar{\theta}
\end{array}\right\}
$$

It can be seen by comparing the above to (9.2.2-3) that when $\bar{\theta}=0$, the above corresponds exactly to the velocity field of classical Mindlin-Reissner theory, and as long as $\bar{\theta}$ is small, it is a good approximation. However, analysts often let $\bar{\theta}$ take on large values, like $\frac{\pi}{4}$, by placing the slave nodes so that the director is not aligned with the normal. When the angle between the director and the normal is large, the velocity field differs substantially from that of classical Mindlin-Reissner theory.

The acceleration is given by the material time derivative of the velocity:

$$
\begin{equation*}
\dot{\mathbf{v}}=\dot{\mathbf{v}}^{M}+\bar{\eta}(\dot{\omega} \times \mathbf{p}+\omega \times(\omega \times \mathbf{p})) \tag{9.4.9}
\end{equation*}
$$

so as indicated in (9.3.17), the accelration depends quadratically on the angular velocities.
The dependent variables for the beam are the two components of the midline velocity, $\mathbf{v}^{M}(\xi, t)$ and the angular velocity $\omega(\xi, t)$; alternatively one can let the midline displacement $\mathbf{u}^{M}(\xi, t)$ and the current angle of the director, $\theta(\xi, t)$, be the dependent variables. Thus the constraints introduced by the assumptions of the CB beam theory change the dependent variables from the two translational velocity components to two translational components and a rotation. However, the new dependent variables are functions of a single space variable, $\xi$, whereas the independent variables of the continuum are functions of two space variables. This reduction in the dimensionality of the problem is the major benefit of structural theories.

The development of expressions for the rate-of-deformation tensor is somewhat involved. The following is based on Belytschko, Wong and Stolarski(1989) specialized to two dimensions. We start with the implicit differentiation formula (4.4.36)

$$
\begin{align*}
& \mathbf{L}=\mathbf{v}_{, \mathbf{x}}=\mathbf{v}_{, \xi} \mathbf{x}_{, \xi}^{-1} \\
& \hat{\mathbf{D}}=\operatorname{sym}\left[\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}\right]=\left[\begin{array}{cc}
\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{x}}-\hat{y} \frac{\partial \omega}{\partial \hat{x}} & \frac{1}{2}\left(\frac{\partial \hat{v}_{y}^{M}}{\partial \hat{x}}-\omega+\frac{\partial \omega}{\partial \hat{x}} \tan \bar{\theta}\right) \\
\operatorname{sym} & \omega \tan \bar{\theta}
\end{array}\right] \tag{9.4.13}
\end{align*}
$$

The effects of deviations of the director from the normal can be seen by comparing the above with (9.2.4). The axial velocity strain, which is predominant in bending response, agrees exactly with the Mindlin-Reissner theory: it varies linearly through the thickness of the beam, with the linear field entirely due to rotation of the cross-section. However, the above transverse shear $\hat{D}_{x y}$ and normal velocity strains $\hat{D}_{y y}$ differ substantially from those of the classical Mindlin-Reissner theory (9.2.4) when the angle $\hat{\theta}$ between the director and the normal to the lamina is large. These differences effect the plane stress
assumption. The motion associated with the modified Mindlin-Reissner theory can generate a significant nonzero axial velocity strain through Poisson effects.

The above tortuous approach is seldom used for the calculation of the velocity strrains in a CB beam. It makes sense only when the nodal internal fores are computed from resultant stresses. Otherwise the standard continuum expressions given in Chapter 4 are utilized. The objective of the above development was to show the characteristics of the velocity strain of a CB beam element, particularly its distribution through the thickness of the beam. The predominantly linear variation of the velocity strains through the thickness is the basis for developing resultant stresses.

Resultant Stresses. In classical beam and shell theories, the stresses are treated in terms of their integrals, known as resultant stresses. In the following, we examine the resultant stresses for CB beam theory, but to make the development more manageable, we assume the director to be normal to the reference surface, i.e. that $\bar{\theta}=0$. We consider a curved beam in two dimensions with the reference line parametrized by $r$; $0 \leq r \leq L$, where $r$ has physical dimensions of length, in contrast to the curvilinear coordinate $\xi$, which is nondimensional. To define the resultant stresses, we will express the virtual internal power (4.6.12) in terms of corotational components of the Cauchy stress. We omit the power due to $\hat{\sigma}_{y y}$, which vanishes due to the plane stress assumption (4.6.12), giving

$$
\begin{equation*}
\delta P^{i n t}=\int_{0}^{L} \int_{A}^{L}\left(\delta \hat{D}_{x} \hat{\sigma}_{x}+2 \delta \hat{D}_{x y} \hat{\sigma}_{x y}\right) d A d r \tag{9.4.13b}
\end{equation*}
$$

In the above, the three-dimensional domain integral has been changed to an area integral and a line integral over the arc length of the reference line. The above integral is exactly equivalent to the integral over the volume if the directors at the endpoints are normal to the reference line. If the directors are not normal to the reference line at the endpoints, then the volume in (9.4.14) differs from the volume of the continuum element as shown in Fig. 9.7. This is usually not significant.


Figure 9.7 Comparison of volume integral in CB beam theory with line integral

Substituting (9.4.13b) into (9.4.13) gives

$$
\begin{align*}
\delta P^{i n t}= & \int_{0}^{L} \int_{A}\left(\frac{\partial\left(\delta \hat{v}_{x}^{M}\right)}{\partial \hat{x}} \hat{\sigma}_{x x}-\frac{\partial(\delta \omega)}{\partial \hat{x}} \hat{y} \hat{\sigma}_{x x}\right. \\
& \left.+\left(-\delta \omega+\frac{\partial\left(\delta \hat{v}_{y}^{M}\right)}{\partial \hat{x}}\right) \hat{\sigma}_{x y}\right) d A d r \tag{9.4.14}
\end{align*}
$$



Figure 9.8. Resultant stresses in 2D beam.

9.9. An example of external loads on a CB beam.

The following area integrals are defined

$$
\begin{array}{ll}
\text { membrane force } & n=\int_{A} \hat{\sigma}_{x x} d A \\
\text { moment } & m=-\int_{A} y \hat{\sigma}_{x x} d A \tag{9.4.15}
\end{array}
$$

shear

$$
s_{\mathrm{y}}=\int_{A} \hat{\sigma}_{x y} d A
$$

The above are known as resultant stresses or generalized stresses; they are shown in Fig. 9.8 in their positive directions. The resultant $n$ is the normal force, also called the
membrane force or axial force. This is the net force tangent to the midline due to the stresses in the beam. The moment $m$ is the first moment of the stresses above the reference line. The shear force $s$ is the net resultant of the transverse shear stresses. These definitions correspond with the customary definitions in texts on structures or mechanics of materials.

With these definitions, the internal virtual power (9.4.14) becomes

$$
\begin{equation*}
\delta P^{i n t}=\int_{0}^{L}(\underbrace{\frac{\partial\left(\delta \hat{v}_{x}^{M}\right)}{\partial \hat{x}} n}_{\text {axial }}+\underbrace{\frac{\partial(\delta \omega)}{\partial \hat{x}} m}_{\text {bending }}+\underbrace{\left(-\delta \omega+\frac{\partial\left(\delta \hat{v}_{y}^{M}\right)}{\partial \hat{x}}\right) q}_{\text {shear }}) d r \tag{9.4.16}
\end{equation*}
$$

The physical names of the various powers are indicated. The axial or membrane power is the power expended on stretching the beam, the bending power the energy expended on bending the beam. The transverse shear power arises also from bending of the beam (see Eq. (???)); it vanishes for thin beams where the Euler-Bernoulli assumption is applicable.

The external power is defined in terms of resultants of the tractions subdivided into axial and bending power in a similar way. We assume $\bar{t}_{z}=0$ and that $\mathbf{p}$ is coincident with $\hat{y}$ at the ends of the beam and consider only the tractions for the specific example shown in Fig. 9.9; the director is assumed collinear with the normal, so only the terms in classical Mindlin-Reissner theory are developed. The virtual external power is obtained from (B4.2.5), which in terms of corotational components gives

$$
\begin{equation*}
\delta P^{e x t}=\int_{\Gamma_{1} \cup \Gamma_{2}}\left(\delta \hat{v}_{x} \hat{t}_{x}^{*}+\delta \hat{v}_{y} \hat{t}_{y}^{*}\right) d \Gamma+\int_{\Omega}\left(\delta \hat{v}_{x} \hat{b}_{x}+\delta \hat{v}_{y} \hat{b}_{y}\right) d \Omega \tag{9.4.17}
\end{equation*}
$$

Substituting Eq. (9.4.12) into the above yields

$$
\begin{align*}
\delta P^{e x t}= & \int_{\Gamma_{1} \cup \Gamma_{2}}\left(\left(\delta \hat{v}_{x}^{M}-\delta \omega \hat{y}\right) \hat{t}_{x}^{*}+\left(\delta \hat{v}_{y}^{M}\right) t_{y}^{*}\right) d \Gamma  \tag{9.4.18}\\
& +\int_{\Omega}\left(\left(\delta \hat{v}_{x}^{M}-\delta \omega \hat{y}\right) \hat{b}_{x}+\left(\delta \hat{v}_{y}^{M}\right) \hat{b}_{y}\right) d \Omega
\end{align*}
$$

The applied forces are now subdivided into those applied to the ends of the beam and those applied over the interior. For this example, only the right hand end is subjected to prescribed tractions, see Fig. 9.9. The generalized external forces are now defined similarly to the resulotant stresses by taking the zeroth and first moments of the tractions:

$$
\begin{equation*}
n^{*}=\int_{\Gamma_{1}} \hat{t}_{x}^{*} d A, \quad s^{*}=\int_{\Gamma_{1}} \hat{t}_{y}^{*} d A, \quad m^{*}=-\int_{\Gamma_{1}} \hat{y}_{x}^{*} d A= \tag{9.4.19}
\end{equation*}
$$

where the last equality follows from the fact that the director is assumed normal to the midline at the boundaries. The tractions between the end points and the body forces are subsumed as generalized body forces

$$
\hat{f}_{x}=\int_{\Gamma_{2}} \hat{t}_{x}^{*} d \Gamma+\int_{\Omega} \hat{b}_{x} d \Omega, \hat{f}_{y}=\int_{\Gamma_{2}} \hat{t}_{y}^{*} d \Gamma+\int_{\Omega} \hat{b}_{y} d \Omega, \quad M=-\int_{\Gamma_{2}} \hat{y}_{x}^{*} d \Gamma+\int_{\Omega} \hat{y}_{y} \hat{b}_{y} d \Omega \text { (9.4.20) }
$$

Since the dependent variables have been changed from $v_{i}(x, y)$ to $v_{i}^{M}(r)$ and $\omega(r)$ by the modified Mindlin-Reissner constraint, the definitions of boundaries are changed accordingly: the boundaries become the end points of the beam. Any loads applied between the endpoints are treated like body forces. The boundaries with prescribed forces are denoted by $\Gamma_{n}, \Gamma_{m}$ and $\Gamma_{s}$ which are the end points at which the normal (axial) force, moment, and shear force are prescribed, respectively. The external virtual power (9.4.17), in light of the definitions (9.4.19-20), becomes

$$
\begin{equation*}
\delta P^{e x t}=\int\left(\delta \hat{v}_{x} \hat{f}_{x}+\delta \hat{v}_{y} \hat{f}_{y}+\delta \omega M\right) d r+\left.\delta \hat{v}_{x} n^{*}\right|_{\Gamma_{n}}+\left.\delta \hat{v}_{y} s^{*}\right|_{\Gamma_{s}}+\left.\delta \omega m^{*}\right|_{\Gamma_{m}} \tag{9.4.21}
\end{equation*}
$$

9.3.?. Boundary Conditions. The velocity (essential) boundary conditions for the CB beam are usually expressed in terms of corotational coordinates so that they have a clearer physical meaning. The velocity boundary conditions are

$$
\begin{array}{lr}
\hat{v}_{x}^{M}=\hat{v}_{x}^{M^{*}} & \text { on } \Gamma_{\hat{v}_{x}} \\
\hat{v}_{y}^{M}=\hat{v}_{y}^{M^{*}} & \text { on } \Gamma_{\hat{v}_{y}}  \tag{9.4.18}\\
\omega=\omega^{*} & \text { on } \Gamma_{\omega}
\end{array}
$$

where the subscript on $\Gamma$ indicates the boundary on which the particular displacement is prescribed. The angular velocity. of course, is independent of the orientation of the coordinate system so we have not superposed hat on it.

The generalized traction boundary conditions are:

$$
\begin{array}{ll}
n=n^{*} & \text { on } \Gamma_{n} \\
s=s^{*} & \text { on } \Gamma_{s}  \tag{9.4.19}\\
m=m^{*} & \text { on } \Gamma_{m}
\end{array}
$$

Note that (9.4.18) and (9.4.19) are sequentially conditions on kinematic and kinetic variables which are conjugate in power. Each pair yields a power, i.e., $n \hat{v}_{x}^{M}$ is the power of the axial force on the boundary, $s \hat{v}_{y}^{M}$ is the power of the transverse force and $m \omega$ is the power of the moment. Since variables which are conjugatge in power can not be prescribed on the same boundary, but one of the pair must be prescribed on any boundary, it follows then that

$$
\begin{align*}
\Gamma_{n} \cup \Gamma_{v_{x}}=\Gamma & \Gamma_{n} \cap \Gamma_{v_{x}}=0 \\
\Gamma_{s} \cup \Gamma_{v_{y}}=\Gamma & \Gamma_{s} \cap \Gamma_{v_{y}}=0  \tag{9.4.20}\\
\Gamma_{m} \cup \Gamma_{v_{\omega}}=\Gamma & \Gamma_{m} \cap \Gamma_{\omega}=0
\end{align*}
$$

So on a boundary point either the moment or rotation, the normal force or the velocity $\hat{v}_{x}^{M}$, the shear or the velocity $\hat{v}_{y}^{M}$ must be prescribed, but no pair can be described on the smae boundary. Even for CB beams, boundary conditions are prescribed in terms of resultants. The velocity boundary conditions can easily be imposed on the nodal degrees of freedom given in (9.3.22), since the midline velocities correspond to the nodal velociities. The traction boundary conditions are

Weak Form. The weak form for the momentum equation for a beam is given by

$$
\begin{equation*}
\delta \mathcal{P}^{\text {inert }}+\delta \mathcal{P}^{\text {int }}=\delta \mathcal{P}^{e x t} \forall\left(\delta v_{x}, \delta v_{y}, \delta \omega\right) \in \mathcal{U}_{0} \tag{9.4.21}
\end{equation*}
$$

where the virtual powers are defined in (9.4.16) and (9.4.21) and $\mathcal{U}_{0}$ is the space of piecewise differentiable functions, i.e. $C^{0}$ functions, which vanish on the corresponding prescribed displacement boundaries. The functions need only be $C^{0}$ since only the first derivatives of the dependent variables appear in the virtual power expressions.

Strong Form. We will not derive the strong form equivalent to (9.4.21) for an arbitrary geometry. This can be done, see Simo and Fox(1989) for example, but it is awkward without curvilinear tensors. Instead, we will develop the strong form for a straight beam of uniform cross-section which lies along the x -axis, with inertia and applied moments neglected. Equation (9.4.21) can then be simplified to

$$
\begin{gather*}
\int_{0}^{L}\left(\delta v_{x, x} n+\delta \omega_{, x} m+\left(\delta v_{y, x}-\delta \omega\right) s-\delta v_{x} f_{x}-\delta v_{y} f_{y}\right) d x  \tag{9.4.22}\\
-\left.\left(\delta v_{x} n^{*}\right)\right|_{\Gamma_{n}}-\left.\left(\delta \omega m^{*}\right)\right|_{\Gamma_{m}}-\left.\left(\delta v_{y} s^{*}\right)\right|_{\Gamma_{s}}=0
\end{gather*}
$$

The hats have been dropped since the local coordinate system coincides with the global system at all points. The procedure for finding the equivalent strong form then parallels the procedure used in Section 4.3. The idea is to remove all derivatives of test functions which appear in the weak form, so that the above can be written as products of the test functions with a function of the resultant forces and their derivatives. This is accomplished by using integration by parts, which is sketched below for each of the terms in the weak form:

$$
\left.\begin{array}{l}
\int_{0}^{L} \delta v_{x, x} n d x=\int_{0}^{L}-\delta v_{x} n_{, x} d x+\left.\left(\delta v_{x} n\right)\right|_{\Gamma_{n}}+\left\langle\delta v_{x} n\right\rangle \\
\int_{0}^{L} \delta \omega_{, x} m d x=\int_{0}^{L}-\delta \omega m_{, x} d x+\left.(\delta \omega m)\right|_{\Gamma_{m}}+\langle\delta \omega m\rangle \\
\int_{0}^{L} \delta v_{y, x} s d x=\int_{0}^{L}-\delta v_{y} s, x  \tag{9.4.25}\\
\\
0
\end{array}\right]+\left.\left(\delta v_{y} s\right)\right|_{\Gamma_{s}}+\left\langle\delta v_{y} s\right\rangle,
$$

In each of the above we have used the fundamental theorem of calculus as given in Section 2.? for a piecewise continuously differentiable function and the fact that the test
functions vanish on the prescribed displacement boundaries, so the boundary term only applies to the complementary boundary points, which are given by (9.4.20). Substituting (9.4.23) to (9.4.25) into (9.4.22) gives

$$
\begin{align*}
& \int_{0}^{L}\left(\delta v_{x}\left(n_{, x}+f_{x}\right)+\delta \omega\left(m_{, x}+s\right)+\delta v_{y}\left(s_{, x}+f_{y}\right)\right) d x+\delta v_{x}^{\prime}(n\rangle+\delta v_{y}\left\langle s^{\prime}\right\rangle+  \tag{9.4.26}\\
& \quad \delta \omega\langle m\rangle+-\left.\delta v_{x}\left(n^{*}-n\right)\right|_{\Gamma_{n}}+\left.\delta \omega\left(m^{*}-m\right)\right|_{\Gamma_{m}}+\left.\delta v_{y}\left(s^{*}-s\right)\right|_{\Gamma_{s}}=0
\end{align*}
$$

Using the density theorem as given in Section 4.3 then gives the following strong form:

$$
\begin{align*}
& n_{, x}+f_{x}=0, \quad s_{, x}+f_{y}=0, \quad m_{, x}+s=0, \\
& \langle n\rangle=0,\langle s\rangle=0,\langle m\rangle=0  \tag{9.4.27}\\
& n=n^{*} \text { on } \Gamma_{n}, \quad s=s^{*} \text { on } \Gamma_{s^{\prime}}, m=m^{*} \text { on } \Gamma_{m}
\end{align*}
$$

which are respectively, the equations of equilibrium, the internal continuity conditions, and the generalized traction (natural) boundary conditions.

The above equilibrium equations are well known in structural mechanics. These equilibrium equations are not equivalent to the continuum equilibrium equations, $\sigma_{i j, j}+b_{i}=0$. Instead, they are a weak form of the continuum equilibrium equations. Their suitability for beams is primarily based on experimental evidence. The error due to the structural assumption can not be bounded rigorously for arbitrary materials. Thus the applicability of beam theory, and by extension the shell theories to be considered later, rests primarily on experimental evidence.

Finite Element Approximation. When the motion is treated in the form (9.4.1) as a function of a single variable, the finite element approximation is constructed by means of one-dimensional shape functions $N_{I}(\xi)$ :

$$
\begin{equation*}
\mathbf{x}(\xi, \eta, t)=\sum_{I=1}^{n_{N}}\left(\mathbf{x}_{I}^{M}(t)+\bar{\eta}_{I} \mathbf{p}_{I}(t)\right) N_{I}(\xi) \tag{9.4.24}
\end{equation*}
$$

As is clear from in the above, the product of the thickness with the director is interpolated. If they are interpolated independently, the second term in the above is quadratic in the shape functions and differs from (9.3.2a). It follows immediately from the above that the original configuration of the element is given by

$$
\begin{equation*}
\mathbf{X}(\xi, \eta)=\sum_{I=1}^{n_{N}}\left(\mathbf{X}_{I}^{M}+\bar{\eta}_{I} \mathbf{p}_{I}^{0}\right) N_{I}(\xi) \tag{9.4.25}
\end{equation*}
$$

The displacement is obtained by taking the difference of (9.4.24) and (9.4.25), which gives

$$
\mathbf{u}(\xi, \eta, t)=\sum_{I=1}^{n_{N}}\left(\mathbf{u}_{I}^{M}(t)+\bar{\eta}_{I}\left(\mathbf{p}_{I}(t)-\mathbf{p}_{I}^{0}\right)\right) N_{I}(\xi)
$$

Taking the material time derivative of the above gives the velocity

$$
\mathbf{v}(\xi, \eta, t)=\sum_{I=1}^{n_{N}}\left(\mathbf{v}_{I}^{M}(t)+\bar{\eta}_{I}\left(\omega \mathbf{e}_{z} \times \mathbf{p}_{I}(t)\right)\right) N_{I}(\xi)
$$

This velocity field is identical to the velocity field generated by substituting (9.3.6) into (9.5.2b). Thus the mechanics of any element generated by this approach will be identical to that of an element implemented directly as a continuum element with the modified Mindlin-Reissner constraints applied only at the nodes, i.e. with the modified MindlinReissner assumptions applied to the discrete equations. Therefore we will not pursue this approach further.


Fig. 9.10 Two-node CB beam element based on 4-node quadrilateral continuum element.

Example 9.1 Two-node beam element. The CB beam theory is used to formulate a 2 -node CB beam element based on a 4 -node, continuum quadrilateral. The element is shown in Fig. 9.10. We place the reference line (midline) midway between the top and bottom surfaces; the line coincides with $\xi=0$ in the parent domain; although this placement is not necessary it is convenient. The master nodes are placed at the intersections of the reference line with the edges of the element. The slave nodes are the
corner nodes and are labeled by the two numbering schemes described previously in Fig. 9.10 .

This motion of the 4 -node continuum element

$$
\begin{equation*}
\mathbf{x}=\sum_{I=1}^{4} x_{\bar{I}}(t) N_{\bar{I}}(\xi, \eta) \tag{E9.1.2}
\end{equation*}
$$

where $N_{i}(\xi, \eta)$ are the standard 4-node isoparametric shape functions

$$
\begin{equation*}
N_{l}(\xi, \eta)=\frac{1}{4}\left(1+\xi_{I} \xi\right)\left(1+\eta_{I} \eta\right) \tag{E9.1.3}
\end{equation*}
$$

The motion of the element when given in terms of one-dimensional shape functions by (9.3.3) is:

$$
\begin{align*}
\mathbf{x}(\xi, \eta, t) & =\mathbf{x}^{M}(\xi, t)+\bar{\eta} \mathbf{p}(\xi, t)  \tag{E9.1.1}\\
& =\mathbf{x}_{1}(t)(1-\xi)+\mathbf{x}_{2}(t) \xi+\bar{\eta} \mathbf{p}_{1}(t)(1-\xi)+\bar{\eta} \mathbf{p}_{2}(t) \xi
\end{align*}
$$

Eqs. (E9.1.1) and (E9.1.3) are equivalent if

$$
\begin{array}{ll}
x_{1}(t)=\frac{1}{2}\left(x_{1}+x_{\overline{2}}\right)=\frac{1}{2}\left(x_{1^{+}}+x_{1^{-}}\right) & x_{2}(t)=\frac{1}{2}\left(x_{\overline{3}}+x_{\overline{4}}\right)=\frac{1}{2}\left(x_{2^{+}}+x_{2^{-}}\right) \\
\mathbf{p}_{1}(t)=\frac{\left(x_{\overline{2}}-x_{\bar{T}}\right) \mathbf{e}_{x}+\left(y_{\overline{2}}-y_{1}\right) \mathbf{e}_{y}}{\left(\left(x_{\overline{2}}-x_{\bar{T}}\right)^{2}+\left(y_{\overline{2}}-y_{1}\right)^{2}\right)^{1 / 2}} & \mathbf{p}_{2}(t)=\frac{\left(x_{\overline{4}}-x_{\overline{3}}\right) \mathbf{e}_{x}+\left(y_{4}-y_{\overline{3}}\right) \mathbf{e}_{y}}{\left(\left(x_{\overline{4}}-x_{\overline{3}}\right)^{2}+\left(y_{4}-y_{\overline{3}}\right)^{2}\right)^{1 / 2}} \tag{E9.1.5a}
\end{array}
$$

Thus the motions given in Eqs. (E9.1.2) and (E9.1.3) are alternate descriptions of
the same motion. Eqs. (E9.1.4) define the location of the master nodes. Eqs. (E9.1.5) define the orientations of the directors.

The degrees of freedom of this CB beam element are

$$
\begin{equation*}
\mathbf{d}^{T}=\left[u_{x 1}, u_{y 1}, \theta_{1}, u_{x 2}, u_{y 2}, \theta_{2}\right] \tag{E9.1.6}
\end{equation*}
$$

where $\theta_{I}$ are the angles between the directors and the $x$-axis measured positively in a counterclockwise direction from the positive x -axis. The nodal velocities are

$$
\begin{equation*}
\dot{\mathbf{d}}^{T}=\left[\dot{u}_{x 1}, \dot{u}_{y 1}, \omega_{1}, \dot{u}_{x 2}, \dot{u}_{y 2}, \omega_{2}\right] \tag{E9.1.7}
\end{equation*}
$$

The nodal forces are conjugate to the nodal velocities in the sense of power, so

$$
\begin{equation*}
\mathbf{f}^{T}=\left[f_{x 1}, f_{y 1}, m_{1}, f_{x 2}, f_{y 2}, m_{2}\right] \tag{E9.1.8}
\end{equation*}
$$

where $m_{I}$ are nodal moments.
The nodal velocities of the slave nodes are next expressed in terms of the master nodal velocities by (9.3.7). The relations are written for each triplet of nodes: a master node and the two associated slave nodes. For each triplet of nodes, the (9.3.7) specialized to the geometry of this example is

$$
\begin{equation*}
\mathbf{v}_{I}^{S}=\mathbf{T}_{I} \mathbf{v}_{I}^{M} \quad(\text { no sum on } I) \tag{E9.1.9}
\end{equation*}
$$

where

$$
\mathbf{v}_{I}^{S}=\left\{\begin{array}{l}
v_{x I^{-}}  \tag{E9.1.10}\\
v_{y I^{-}} \\
v_{x I^{+}} \\
v_{y I^{+}}
\end{array}\right\}, \quad \mathbf{T}_{I}=\left[\begin{array}{ccc}
1 & 0 & \frac{h}{2} p_{x} \\
0 & 1 & -\frac{h}{2} p_{y} \\
1 & 0 & -\frac{h}{2} p_{x} \\
0 & 1 & \frac{h}{2} p_{y}
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & \frac{1}{2} y_{12} \\
0 & 1 & \frac{1}{2} x_{2 \mathrm{~T}} \\
1 & 0 & \frac{1}{2} y_{34} \\
0 & 1 & \frac{1}{2} x_{\overline{43}}
\end{array}\right], \quad \mathbf{v}_{I}^{M}=\left\{\begin{array}{c}
v_{x I} \\
v_{y I} \\
\theta_{I}
\end{array}\right\}
$$

Once the slave node velocities are known, the rate-of-deformation can be computed at any point in the element by Eq. (E4.2.c).

The rate-of-deformation is be computed at all quadrature points in the corotational coordinate system of the quadrature point. The two node element avoids shear locking if a single stack of quadrature points $\xi=\left(0, \eta_{Q}\right), Q=1$ to $n_{Q}$. The strain measures are computed in the global coordinate system using the equation given in Example 4.2 and 4.10.

The constitutive equation is evaluated at the quadrature points of the element in a corotational coordinate system given by Eq. (9.3.9) with

$$
\begin{equation*}
\hat{\mathbf{e}}_{x}=\frac{x_{\xi} \mathbf{e}_{x}+y_{, \xi} \mathbf{e}_{y}}{\left(\left(x_{\xi}\right)^{2}+\left(y_{, \xi}\right)^{2}\right)^{\frac{1}{2}}} \quad \hat{\mathbf{e}}_{y}=\hat{\mathbf{e}}_{z} \times \hat{\mathbf{e}}_{x} \tag{E9.1.11}
\end{equation*}
$$

where

$$
\begin{equation*}
x_{, \xi}=\sum_{I=1}^{4} x_{\bar{I}} N_{\bar{I}, \xi} \quad y_{, \xi}=\sum_{I=1}^{4} y_{\bar{I}} N_{\bar{I}, \xi} \tag{E9.1.13}
\end{equation*}
$$

A hypoelastic law for isotropic and anisotropic laws is given by (9.3.11) or (9.3.13), respectively.

The internal forces are then transformed to the master nodes for each triplet by (4.5.36). This gives

$$
\left\{\begin{array}{l}
f_{x I}  \tag{E9.1.14}\\
f_{y I} \\
m_{I}
\end{array}\right\}=\mathbf{T}_{I}^{T H}\left\{\begin{array}{l}
f_{x I^{+}} \\
f_{y I^{+}} \\
f_{x I^{-}} \\
f_{y I}
\end{array}\right\}
$$

Evaluating the first and third term of the above left hand matrix gives

$$
\begin{array}{ll}
f_{x I}=f_{x I^{+}}+f_{x I^{-}} & f_{y I}=f_{y I^{+}}+f_{y I^{-}} \\
m_{1}=\frac{1}{2}\left(y_{\mathrm{T} \mathrm{I}} f_{x 1}+x_{\overline{2} \bar{T}} f_{y 1}\right) & \tag{E.9.1.15b}
\end{array}
$$

So the transformation gives what is expected from equilibrium of the slave node with the master node. The master node force is the sum of the slave node forces and the master node moment is the moment of the slave node forces about the master node.

This element formulation can also be applied to constitutive equations in terms of the PK2 stress and the Green strain. The computation of the Green strain tensors requires the knowledge of $\theta_{I}$ and $x_{I}$. The director in the initial and current configurations is given by

$$
\begin{equation*}
p_{x I}^{0}=\cos \theta_{I}^{0}, \quad p_{y I}^{0}=\sin \theta_{I}^{0} \quad p_{x I}=\cos \theta_{I}, \quad p_{y I}=\sin \theta_{I} \tag{E9.1.11}
\end{equation*}
$$

The positions of the slave nodes can then be computed by specializing (9.4.1) to the nodes, which gives

$$
\begin{array}{llll}
X_{1}=X_{1}+\frac{h}{2} p_{x 1}, & Y_{\mathrm{T}}=Y_{1}+\frac{h}{2} p_{y 1}^{0} & x_{1}=x_{1}+\frac{h}{2} p_{x 1}, & y_{\mathrm{T}}=y_{1}+\frac{h}{2} p_{y 1} \\
X_{\overline{2}}=X_{1}-\frac{h}{2} p_{x 1}^{0}, & Y_{\overline{2}}=Y_{1}-\frac{h}{2} p_{y 1}^{0} & x_{\overline{2}}=x_{1}-\frac{h}{2} p_{x 1}, & y_{\overline{2}}=y_{1}-\frac{h}{2} p_{y 1} \\
X_{\overline{3}}=X_{2}-\frac{h}{2} p_{x 2}^{0}, & Y_{\overline{3}}=Y_{2}-\frac{h}{2} p_{y 2}^{0} & x_{\overline{3}}=x_{2}-\frac{h}{2} p_{x 2}, & y_{\overline{3}}=y_{2}-\frac{h}{2} p_{y 2}  \tag{E9.1.12}\\
X_{4}=X_{2}+\frac{h}{2} p_{x 2}^{0}, & Y_{4}=Y_{2}+\frac{h}{2} p_{y 2}^{0} & x_{\overline{4}}=x_{2}+\frac{h}{2} p_{x 2}, & y_{\overline{4}}=y_{2}+\frac{h}{2} p_{y 2}
\end{array}
$$

The displacement of the slave nodes is then obtained by taking the difference of the nodal coordinates. The displacement of any point can then be obtained by the continuum displacement field

$$
\mathbf{u}=\sum_{I=1}^{4} \mathbf{u}_{\bar{I}} N_{\bar{I}}
$$

The Green strain can then be computed by (3.3.6) and the PK2 stress by the constitutive law. After transforming the PK2 stress to the Cauchy stress by Box 3.2, the nodal forces can be computed as before.

Velocity Strains for Rectangular Element. When the underlying continuum element is rectangular (because the directors are in the $y$ direction), and the beam is along the x direction, the velocity field (9.4.8) is

$$
\mathbf{v}=\mathbf{v}^{M}-y \omega \mathbf{e}_{x}
$$

where we have specialized Eq. (9.4.8) to $\theta=\pi / 2$. Writing out the components of the above and immediately substituting the one-dimensional two-node shape functions gives

$$
\begin{aligned}
& v_{x}=v_{x 1}^{M}(1-\xi)+v_{x 2}^{M} \xi-y\left(\omega(1-\xi)-\omega_{2} \xi\right) \\
& v_{y}=v_{y 1}^{M}(1-\xi)+v_{y 2}^{M} \xi
\end{aligned}
$$

The velocity strain is then given by Eq. (3.3.10):

$$
\begin{aligned}
& D_{x x}=\frac{\partial v_{x}}{\partial x}=\frac{1}{\ell}\left(v_{x 2}^{M}-v_{x 1}^{M}\right)-\frac{y}{\ell}\left(\omega_{2}-\omega_{1}\right) \\
& 2 D_{x y}=\frac{\partial v_{y}}{\partial x}+\frac{\partial v_{x}}{\partial y}=\frac{1}{\ell}\left(v_{y 2}^{M}-v_{y 1}^{M}\right)-\left(\omega_{1}(1-\xi)+\omega_{2} \xi\right) \\
& D_{y y}=0
\end{aligned}
$$

The material tangent and goemetric stiffness of this elementis given by LIU-give result with some explanation

### 9.5 CONTINUUM BASED SHELL IMPLEMENTATION

In this Section, the degenerated continuum (CB) approach to shell finite elements is developed. This approach was pioneered by Ahmad(1970); a nonlinear version of this theory was presented by Hughes and Liu(1981). In the CB approach to shell theory, as for CB beams, it is not necessary to develop the complete formulation, i.e. developing a weak form, discretizing the problem by using finite elemeny interpolatns, etc. Instead the shell element is developed in this Section by imposing the constraints pf the shell theory on a continuum element. Subsequently, we will examine CB shells from a more theoretical viewpoint by imposing the constraints on thhe test and trial functions prior to construction of the weak form.

Assumptions in Classical Shell Theories. To describe the kinematic assumptions for shells, we need to define a reference surface, often called a midsurface. The reference surface, as the second name implies, is generally placed midway between the top and bottom surfaces of the shell. As in nonlinear beams, the exact placement of the reference surface in nonlinear shells is irrelevant.

Before developing the CB shell theory, we briefly review the kinematic assumptions of classical shell theories. Similar to beams, there are two types of kinematic assumptions, those that admit transverse shear and those that don't. The theory which admit transverse shear are called Mindlin-Reissner theories, whereas the theory which does not admit transverse shear is called Kirchhoff-Love theory. The kinematic assumptions in these shell theories are:

1. the normal to the midsurface remains straight (Mindlin-Reissner theory).
2. the normal to the midsurface remains straight and normal (Kirchhoff-Love theory)

Experimental results show that the Kirchhoff-Love assumptions are the most accurate in predicting the behavior of thin shells. For thicker shells, the Mindlin-Reissner assumptions are more accurate because transverse shear effects become important. Transverse shear effects are particularly important in composites. Mindlin-Reissner theory can also be used for thin shells: in that case the normal will remain approximately normal and the transverse shears will almost vanish.

One point which needs to be made is that these theories were originally developed for small deformation problems, and most of their experimental verification has been made for small strain cases. Once the strains are large, it is not clear whether it is better to assume that the current normal remains straight or that the initial normal remains straight. Currently, in most theoretical work, the initial normal is assumed to remain straight. This choice is probably made because it leads to a cleaner theory. We know of no experiments that show an advantage of this assumption over the assumption that the current normal remain instantaneously straight.

Degenerated Shell Methodology. In the implementation and theory of CB shell elements, the shell is modeled by a single layer of three dimensional elements, as shown in Fig. 9.11?. The motion is then constrained to reflect the modified Mindlin-Reissner assumptions.

We consider a shell element, such as the one shown in Fig. 9.11, which is associated with a three dimensional continnjum element. The parent element coordinates are $\xi_{i}$, $i=1$ to 3 ; we also use the notation $\xi_{1} \equiv \xi, \xi_{2} \equiv \eta$, and $\xi_{3} \equiv \zeta$. In the shell, the coordinates $\xi_{i}$ are curvilinear coordinates. The midsurface is the surface given by $\zeta=0$. Each surface of constant $\zeta$ is called a lamina. The reference surface is parametrized by the two curvilinear coordinates $(\xi, \eta)$ or $\xi_{\alpha}$ in indicial notation (Greek letters are used for indices with a range of 2 ). Lines along the $\zeta$ axis are called fibers, and the unit vector along a fiber is called a director. These definitions are analogous to the corresponding definitions for beams given previously.

In the CB shell theory, the major assumptions are the modified Mindlin-Reissner kinematic assumption and the plane stress assumption:

1. fibers remain straight;
2. the stress normal to the midsurface vanishes.

Often it is assumed that the fibers are inextensional but we omit this assumption. These assumptions differs from those of classical Mindlin-Reissner theory in that the rectilinear constraint applies to fibers, not to the normals. This modification is chosen because, as in beams, the Mindlin-Reissner kinematic assumption cannot be imposed exactly in a CB element with $C^{0}$ interpolants. In models based on the modified Mindlin-Reissner theory, the nodes should be placed so that the fiber direction is as close as possible to normal to the midsurface.

For thin shells, the behavior of CB shells will approximate the behavior of a Kirchhoff-Love shell: normals to the midsurface will remain normal, so directors which are originally normal to the midsurface will remain normal, and the transverse shears will vanish. The normality constraint is based on physical observations, and even when this constraint is not imposed on a numerical model, the results will tend towards this behavior for thin shells.

We will consider shells where the deformations are large enough so that the thickness may change substantially with deformation. The thickness change arises from the conservation of matter, but is usually imposed on the model through the constitutive equations, which reflect the conservation of matter. In order to model the thickness change exactly, it is necessary to integrate the thickness strains along the entire fiber. Here we present a simpler and computationally less demanding theory which only accounts for a linear variation of the thickness strain through the depth of the shell. This is more accurate than theories which incorporate only the overall thickness change and is usually very accurate, since the major effect which needs to be modeled, in addition to the thickness change due to elongational straining, are the consequences of the linear bending field.

The motion of the shell is given by

$$
\begin{align*}
& \mathbf{x}(\xi, \eta, \zeta, t)=\mathbf{x}^{M}(\xi, \eta, t)+\zeta h^{-} \mathbf{p}(\xi, \eta, t) \text { for } \zeta<0  \tag{9.5.1a}\\
& \mathbf{x}(\xi, \eta, \zeta, t)=\mathbf{x}^{M}(\xi, \eta, t)+\zeta h^{+} \mathbf{p}(\xi, \eta, t) \text { for } \zeta>0 \tag{9.5.1b}
\end{align*}
$$

where $h^{-}$and $h^{+}$are the distances from the midsurface to the top and bottom surfaces along the director, respectively. The above will be written in the compact form

$$
\begin{equation*}
\mathbf{x}=\mathbf{x}^{M}+\bar{\zeta} \mathbf{x}^{B} \tag{9.5.2}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{x}^{B}=\mathbf{p}  \tag{9.5.2b}\\
& \bar{\zeta}=\zeta h^{+} \text {when } \zeta>0, \quad \bar{\zeta}=\zeta h^{-} \text {when } \zeta<0 \tag{9.5.3}
\end{align*}
$$

In the above, $\mathbf{x}^{B}$ characterizes the motion due to bending; although this decomposition of the motio, it becomes more useful for other kinematic quantities.

The coordinates of the shell in the original configuration are obtained by evaluating (9.5.2) at the initial time

$$
\begin{equation*}
\mathbf{X}(\xi, \eta, \zeta)=\mathbf{X}^{M}(\xi, \eta)+\bar{\zeta} \mathbf{p}_{0}(\xi, \eta)=\mathbf{X}^{M}+\bar{\zeta} \mathbf{X}^{B} \tag{9.5.4}
\end{equation*}
$$

where $\mathbf{p}_{0}=\mathbf{p}(\xi, \eta, 0)$. The displacement field is obtained by taking the difference of (9.5.2) and (9.5.4):

$$
\begin{equation*}
\mathbf{u}(\xi, \eta, \zeta, t)=\mathbf{u}^{M}+\bar{\zeta}\left(\mathbf{p}-\mathbf{p}_{0}\right)=\mathbf{u}^{M}+\bar{\zeta} \mathbf{u}^{B} \tag{9.5.5a}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{u}^{M}=\mathbf{x}^{M}-\mathbf{X}^{M} \quad \mathbf{u}^{B}=\mathbf{p}-\mathbf{p}_{0} \tag{9.5.5b}
\end{equation*}
$$

As can be seen from the above, the bending displacement field $\mathbf{u}^{B}$ is the difference between two unit vectors. Therefore the bending field can be described by two dependent
variables. The motion is then described by 5 dependent variables: the three translations of the midsurface, $\mathbf{u}^{M}=\left[u_{x}^{M}, u_{y}^{M}, u_{z}^{M}\right]$ and the two dependent variables which describe the bending displacement, $\mathbf{u}^{B}$, which remain to be defined.

The velocity field is obtained by taking the material time derivative of the displacement or motion, using (9.2.1) to write the rate of the director:

$$
\begin{equation*}
\mathbf{v}(\xi, \eta, \zeta, t)=\mathbf{v}^{M}(\xi, \eta, t)+\bar{\zeta} \omega(\xi, \eta, t) \times \mathbf{p}+\dot{\bar{\zeta}} \mathbf{p} \tag{9.5.6}
\end{equation*}
$$

The last term in the above represents the change in thickness of the shell. It will not be retained in the equations of motion, since it represents an insignificant inertia. But it will be used in updating the geometry, so it will effect the internal nodal forces, which depend on the current geometry. The variable $\dot{\bar{\zeta}}$ will be obtained from the constitutive equation or conservation of matter. The velocity field can also be written as

$$
\begin{equation*}
\mathbf{v}(\xi, \eta, \zeta, t)=\mathbf{v}^{M}+\bar{\zeta} \mathbf{v}^{B}+\dot{\bar{\zeta}} \mathbf{p} \tag{9.5.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{v}^{B}=\bar{\zeta} \omega \times \mathbf{p} \tag{9.5.8}
\end{equation*}
$$

As can be seen from the above, the velocity of any point in the shell consists of the sum of the velocity of the reference plane, the bending velocity, and the velocity due to the change in thickness. The bending velocity is defined by the rotation of the director. Only the two components of the angular velocity in the plane tangent to the director $\mathbf{p}$ are relevant. The component parallel to the $\mathbf{p}$ vector is irrelevant since it causes no change in the director $\mathbf{p}$. This component is called the drilling component or the drill for short.

Local and Corotational Coordinates. Three coordinate systems are defined:

1. the global Cartesian system, $(x, y, z)$ with base vectors $\mathbf{e}_{i}$.
2. the corotational Cartesian coordinates $(\hat{x}, \hat{y}, \hat{z})$ with base vectors $\hat{\mathbf{e}}_{i}$, which are constructed so that the plane defined by $\hat{\mathbf{e}}_{1}(\xi, t)$ and $\hat{\mathbf{e}}_{2}(\xi, t)$ is tangent to the lamina As indicated, the corotational base vectors are functions of the element coordinates and time. In practice, these coordinate systems are constructed only at the quadrature points of the element, but conceptually, the corotational coordinate system is defined at every point of the shell. Several methods have been proposed for the construction of the corotational systems, and they will be described later.
3. nodal coordinate systems associated with the master nodes; they are denoted by superposed bars $\overline{\mathbf{e}}_{i I}(t)$, where the subscript the node. The nodal coordinates system is defined by

$$
\begin{equation*}
\overline{\mathbf{e}}_{z I}(t)=\mathbf{p}_{I}(t) \tag{9.5.9}
\end{equation*}
$$

The orientation of the two other base vectors is described later.

Finite Element Approximation of Motion. The underlying finite element for a CB shell theory is a three-dimensional isoparametric element with $2 n_{N}$ slave nodes. In order to meet the modified Mindlin-Reissner assumption, the continuum element may have at most two slave nodes along any fiber. As a consequence of this restriction, the motion will be linear in $\zeta$. The description is Lagrangian and either an updated or total Lagrangian formulation can be employed. We will emphasize the updated Lagrangian formulation, but remind the reader that in the updated Lagrangian formulation the strain can be described by the Green strain tensor and the PK2 stress when it is advantageous for a particular constitutive law. Moreover any updated Lagrangian formulation can easily be changed to a Lagrangian formulation by a transformation of stresses and change in the domain of integration.

The formulation may have either 5 or 6 degrees of freedom per master node. We will emphasize the 5 degree-of-freedom formulations and discuss the relative merits later. The degrees of freedom in the 5 degree-of-freedom formulation are

$$
\begin{equation*}
\mathbf{v}_{I}=\left[v_{x I}, v_{y I}, v_{z I}, \bar{\omega}_{x I}, \bar{\omega}_{y I}\right]^{T} \tag{9.5.9b}
\end{equation*}
$$

the $\bar{\omega}_{z I}$ component, the drilling angular velocity component, has been omitted; see (9.5.9) for the definition of the nodal coordinate system. The nodal forces are conjugate to the nodal velocities in the sense of power, so they are given by

$$
\begin{equation*}
\mathbf{f}_{I}=\left[f_{x I}, f_{y I}, f_{z I}, \bar{m}_{x I}, \bar{m}_{y I}\right]^{T} \tag{9.5.9c}
\end{equation*}
$$

At the intersections of the slave nodal fibers with the reference surface, we define master nodes as shown in Fig. 9.7. The finite element approximation to the motion in terms of the motion of the slave nodes is

$$
\begin{equation*}
\mathbf{x}\left(\xi^{e}, t\right) \equiv \phi^{h}\left(\xi^{e}, t\right)=\sum_{I=1}^{2 n_{N}} \mathbf{x}_{I}(t) N_{\bar{I}}\left(\xi^{e}\right)=\sum_{I^{+}=1}^{n_{N}} \mathbf{x}_{I^{+}}(t) N_{I^{+}}\left(\xi^{e}\right)+\sum_{I^{-}=1}^{n_{N}} \mathbf{x}_{I^{-}}(t) N_{I^{-}}\left(\xi^{e}\right) \tag{9.5.10}
\end{equation*}
$$

where $N_{\bar{I}}\left(\xi^{e}\right)$ are standard isoparametric, three dimensional shape functions and $\xi^{e}$ are the parent element coordinates. Recall that in a Lagrangian element, the element coordinates can be used as surrogate material coordinates. The above gives the motion for a single element; the assembly of element motions to obtain the motion of the complete body is standard.

Two notations are used for the slave nodes: nodal indices with superposed bars, which refer to the original node numbers of the underlying three dimensional element and node numbers with plus and minus superscripts, which refer to the master node numbers. Nodes $I^{+}$and $I^{-}$are, respectively, the slave nodes on the top and bottom surfaces of the fiber which passes through master node $I$.

The velocity field of the underlying continuum element is given by

$$
\begin{equation*}
\mathbf{v}\left(\xi^{e}, t\right)=\frac{\partial \phi^{h}\left(\xi^{e}, t\right)}{\partial t}=\sum_{i=1}^{2 n_{N}} \dot{\mathbf{x}}_{l}(t) N_{l}\left(\xi^{e}\right) \tag{9.5.12}
\end{equation*}
$$

where $\dot{\mathbf{x}}_{I}$ is the velocity of slave node $I$. To achieve a velocity field compatible with (9.5.6), the velocity of the slave nodes is given in terms of the translational velocities of the master nodes $\mathbf{v}_{I}^{M}=\left[v_{x I}^{M}, v_{y I}^{M}, v_{z I}^{M}\right]^{T}$ and the angular velocities of the director $\omega_{I}=\left[\bar{\omega}_{x I}, \bar{\omega}_{y I}\right]^{T}$ by

$$
\begin{aligned}
& \mathbf{v}_{I^{+}}=\mathbf{v}_{I}^{M}+h^{+} \omega_{I} \times \mathbf{p}_{I}-\dot{h}_{I}^{+} \mathbf{p}_{I} \quad \mathbf{v}_{I^{-}}=\mathbf{v}_{I}^{M}+h_{I}^{-} \omega_{I} \times \mathbf{p}_{I}+\dot{h}_{I}^{-} \mathbf{p}_{I} \\
& h=\int_{0}^{1} h_{0} F_{\zeta \zeta} d \zeta
\end{aligned}
$$

where $\dot{h}_{I}^{+}$and $\dot{h}_{I}^{-}$are the velocities of slave nodes $I^{+}$and $I^{-}$in the direction of the director, respectively. These are obtained from integrating the through-the thickness strains obtained from the constitutive equation because of the plane stress assumption, as described later. They are omitted in the formulation of the equations of motion, for neither momentum balance nor equilibrium is enforced in the direction of $\mathbf{p}$.

The relationship between the slave and master nodal velocities for each triplet of nodes along a fiber can then be written in matrix form as

$$
\left\{\begin{array}{l}
\overline{\mathbf{v}}_{I^{+}}  \tag{9.5.14}\\
\overline{\mathbf{v}}_{I^{-}}
\end{array}\right\}=\mathbf{T}_{I} \overline{\mathbf{v}}_{I} \quad \text { no sum on } I
$$

where the vector have been expressed in the nodal coordinate system of the master node for convenience. For a 5 degree of freedom per node formulation

$$
\begin{align*}
& \overline{\mathbf{v}}_{I^{+}}=\left[\bar{v}_{x I}, \bar{v}_{y I^{+}}, \bar{v}_{z I^{+}}\right]^{T} \quad \overline{\mathbf{v}}_{I^{-}}=\left[\bar{v}_{x I^{-}}, \bar{v}_{y I^{-}}, \bar{v}_{z I^{-}}\right]^{T}  \tag{9.5.15}\\
& \overline{\mathbf{v}}_{I}=\left[\bar{v}_{x I}, \bar{v}_{y I}, \bar{v}_{z I}, \bar{\omega}_{x I}, \bar{\omega}_{y I}\right]  \tag{9.5.16}\\
& \mathrm{T}_{I}=\left[\begin{array}{ll}
\mathbf{I}_{3 \times 3} & \Lambda^{+} \\
\mathbf{I}_{3 \times 3} & \Lambda^{-}
\end{array}\right]  \tag{9.5.17}\\
& \Lambda^{+}=h_{I}^{+}\left[\begin{array}{cc}
0 & 1 \\
-1 & 0 \\
0 & 0
\end{array}\right] \quad \Lambda^{-}=h_{I}^{-}\left[\begin{array}{cc}
0 & 1 \\
-1 & 0 \\
0 & 0
\end{array}\right] \tag{9.5.18}
\end{align*}
$$

For a 6 degree-of-freedom per master node formulation it is more convenient to write (9.5.14) in terms of global components:

$$
\left\{\begin{array}{c}
\mathbf{v}_{I^{+}}  \tag{9.5.19}\\
\mathbf{v}_{I^{-}}
\end{array}\right\}=\mathbf{T}_{I} \mathbf{v}_{I} \quad \text { no sum on } I
$$

$$
\begin{align*}
& \mathbf{v}_{I^{+}}=\left[v_{x I}, v_{y I^{+}}, v_{z I^{+}}\right]^{T} \quad \mathbf{v}_{I^{-}}=\left[v_{x I^{-}}, v_{y I^{-}}, v_{z I^{-}}\right]^{T}  \tag{9.5.20}\\
& \mathbf{v}_{I}=\left[v_{x I}, v_{y I}, v_{z I}, \omega_{x I}, \omega_{y I}, \omega_{z I}\right]  \tag{9.5.22}\\
& \Lambda^{+}=h_{I}^{+}\left[\begin{array}{ccc}
0 & p_{z} & -p_{y} \\
-p_{z} & 0 & p_{x} \\
p_{y} & -p_{x} & 0
\end{array}\right]=\left[\begin{array}{ccc}
0 & z_{I+}-z_{I} & y_{I}-y_{I+} \\
z_{I}-z_{I+} & 0 & x_{I+}-x_{I} \\
y_{I+}-y_{I} & x_{I}-x_{I+} & 0
\end{array}\right]  \tag{9.5.21}\\
& \Lambda^{-}=-h_{I}^{-}\left[\begin{array}{ccc}
0 & p_{z} & -p_{y} \\
-p_{z} & 0 & p_{x} \\
p_{y} & -p_{x} & 0
\end{array}\right]=\left[\begin{array}{ccc}
0 & z_{I-}-z_{I} & y_{I}-y_{I-} \\
z_{I}-z_{I-} & 0 & x_{I-}-x_{I} \\
y_{I-}-y_{I} & x_{I}-x_{I-} & 0
\end{array}\right] \tag{9.5.23}
\end{align*}
$$

Nodal Internal Forces. The nodal forces at the slave nodes, i.e. the nodes of the underlying continuum element, are obtained by the usual procedures for continuum elements, see Chapter 4. Of course, the plane stress assumption and computation of the thickness change must be considered in the procedures at the continuum level.

The nodal internal and external forces at the master nodes can be obtained from the slave nodal forces by Eq. (4.3.36), which using (9.5.14) gives

$$
\mathbf{f}_{I}=\mathbf{T}_{I}\left\{\begin{array}{l}
\mathbf{f}_{I^{+}}  \tag{9.5.24}\\
\mathbf{f}_{\Gamma}
\end{array}\right\} \quad \text { no sum on } I
$$

where for a 6 degree-of-freedom formulation

$$
\begin{equation*}
\mathbf{f}_{I}=\left[f_{x I}, f_{y I}, f_{z I}, m_{x I}, m_{y I}, m_{z I}\right] \tag{9.5.25}
\end{equation*}
$$

and $\mathbf{T}_{I}$ is given by (9.5.19-23). In the above, $m_{i l}$ are the nodal moments at the master nodes.

Tangent Stiffness. The tangent stiffness matrix can be obtained from that of the underlying continuum element by the standard transformation for stiffness matrices, Section

$$
\begin{equation*}
\mathbf{K}_{I J}=\mathbf{T}_{I}^{T} \overline{\mathbf{K}}_{I J} \mathbf{T}_{J} \quad \text { no sum on I or J } \tag{9.5.26}
\end{equation*}
$$

where $\overline{\mathbf{K}}_{I J}$ is the tangent stiffness matrix for the continuum element.
The rate-of-deformation is computed in the corotational coordinates system with base vectors $\hat{\mathbf{e}}_{i}$. The equations for the rate-of-deformation in the corotational coordinates, ; are

$$
\hat{D}_{i j}=\frac{1}{2}\left(\frac{\partial \hat{v}_{i}}{\partial \hat{x}_{j}}+\frac{\partial \hat{v}_{j}}{\partial \hat{x}_{i}}\right)
$$

The rate-of-deformation $\hat{D}_{z z}$ is computed from the conservation of mass or the condition that the normal stress $\hat{\sigma}_{z z}$ vanishes. This is discussed in more detail in Section ??.

Applying these equations to the velocity field (9.5.6-7) gives

$$
\begin{aligned}
& \hat{D}_{x x}=\frac{\partial \hat{v}_{x}}{\partial \hat{x}}=\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{x}}+\zeta \frac{\partial \hat{v}_{x}^{B}}{\partial \hat{x}} \approx \\
& \hat{D}_{y y}=\frac{\partial \hat{v}_{y}}{\partial \hat{y}}=\frac{\partial \hat{v}_{y}^{M}}{\partial \hat{y}}+\zeta \frac{\partial \hat{v}_{y}^{B}}{\partial \hat{y}} \approx \\
& \hat{D}_{x y}=\frac{1}{2}\left(\frac{\partial \hat{v}_{x}}{\partial \hat{y}}+\frac{\partial \hat{v}_{y}}{\partial \hat{x}}\right)=\frac{1}{2}\left(\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{y}}+\frac{\partial \hat{v}_{y}^{M}}{\partial \hat{x}}\right)+\frac{1}{2} \zeta\left(\frac{\partial \hat{v}_{x}^{B}}{\partial \hat{y}}+\frac{\partial \hat{v}_{y}^{B}}{\partial \hat{x}}\right) \approx \\
& \hat{D}_{x z}=\frac{1}{2}\left(\frac{\partial \hat{v}_{x}}{\partial \hat{z}}+\frac{\partial \hat{v}_{z}}{\partial \hat{x}}\right)=\frac{1}{2}\left(\frac{\partial \hat{v}_{y}^{M}}{\partial \hat{x}}+\zeta \frac{\partial \hat{v}_{y}^{B}}{\partial \hat{x}}\right) \approx
\end{aligned}
$$

In deriving the above, we have used the fact that the tangent plane to the lamina is coincident with the $\hat{x}, \hat{y}$ plane, so functions of $\xi$ and $\eta$ are independent of $\hat{z}$. The above equations are very similar to the equations we derived for a plate, Exercise ??. However, it is implicit in Eqs.(??) that the $\hat{x}, \hat{y}$ plane is constructed so that it is tangent to the lamina which passes through the point at which the rate-of-deformation is evaluated. As a consequence additional terms appear in the actual rate-of-deformation fields; these are explored in Example ???.

The $\hat{D}_{x x}, \hat{D}_{y y}$ and $\hat{D}_{x y}$ components of the rate-of-deformation consist of a membrane part that is constant through the depth of the shell and a bending part which varies linearly through the depth of the shell. The transverse shears $\hat{D}_{x z}$ and $\hat{D}_{y z}$ are constant through the thickness. This characteristic of the transverse shears does not agree with actual behavior of shells and is dealt with in many cases by a shear correction factor.

Discrete momentum equation. The discrete equations for the shell are obtained via the principle of virtual power. As mentioned before, the only difference in the way the principle of virtual power is applied to a shell element is that the kinematic constraints are taken into account. We will use the same systematic procedure as before of identifying the virtual power terms by the physical effects from which they arise and then developing corresponding nodal forces. The main difference we will see is that in the shell theory nodal moments arise quite naturally, so we will treat the nodal moments separately. If the angular velocity and the director are expressed in terms of shape functions, the product of shape functions will not be compatible with the reference
continuum element and the result will not satisfy the reproducing conditions for linear polynomials.

Inconsistencies and Idiosyncrasies of Structural Theories. The introduction of both the Mindlin-Reissner and Kirchhoff-Love assumptions introduces several inconsistencies into the resulting theory. In the Mindlin-Reissner theory, the shear stresses $\hat{\sigma}_{x z}$ and $\hat{\sigma}_{y z}$ are constant through the depth of the shell. However, unless a shear traction is applied to the top or bottom surfaces, the transverse shear must vanish at these surfaces because of the symmetry of the stress tensor. Furthermore, a simple analysis of the requirements of equilibrium in a beam shows that the transverse shear stress are quadratic through the depth of a beam, vanishing at the top and bottom surfaces. Therefore a constant shear stress distribution overestimates the shear energy . A correction factor, knownas a shear correction, is often used on the transverse shear to reduce the energy associated with it, and accurate estimates of this factor can be made for elastic beams and shells. For nonlinear materials, however, it is difficult to estimate a shear correction factor.

The inconsistency of Kirchhoff-Love theory is even more drastic, since the kinematic assumption results in a vanishing transverse shear. In a beam, it is well known in structural theories that the shear must be nonzero if the moment is not constant. Thus the Kirchhoff-Love kinematic assumption is quite inconsistent with equilibrium. Nevertheless, comparison with experiments shows that it is quite accurate, and for thin, homogeneous shells it is more effective and just as accurate as the Mindlin-Reissner theory. Transverse shear simply does not play an important role in the deformation of thin structures, so its inclusion has little effect, but Mindlin-Reissner theories are nevertheless used in finite elements because of the simplicity of the CB shell approach.

The use of the modified Mindlin-Reisner CB models pose additional possibilities for severe errors. If the directors are not normal to the midsurface, the motion deviates markedly from the motion which has been verified experimentally for thin and thick beams and shells. Bathe shows results for CB shells elements modeling a frame with a right angle corner which are at least reasonable. However, when a right angle is included in the model, the assumption that the fiber direction be near to the normal to the midsurface obviously no longer holds. In view of this, it would be foolhardy to use CB elements without modifying the construction of the director in the vicinity of sharp corners.

The zero normal stress, i.e. the plane stress, assumption is also inconsistent when a normal traction is applied to either surface of the shell. Obviously, the normal stress must equal the applied normal traction for equilibrium. However, it is neglected in structural theories because it is much smaller than the axial stresses, so the energy associated with it is much smaller and it has little effect on the deformation.

Another effect of which the analyst should remain aware is boundary effects in shells. Certain boundary conditions result in severe edge effects where the behavior changes dramatically in a narrow boundary layer. The standard boundary conditions also can result in singularities at corners, (MORE DETAIL)

An important reason for using the structural kinematic assumptions is that they improve the conditioning of the discrete equations. If a shells is modeled with three-dimensional continuum elements, the degrees of freedom are the translations at all of the nodes. The mode associated with through-the-thickness velocity strains $\hat{D}_{z z}$ then has very large eigenvalues, so the conditioning of the equations is very poor. The conditioning of shell
equations is also not as good as that of standard continuum models, but it is substantially better than that of continuum models of thin shells.


Fig. 9.?? Rotation of a vector $\mathbf{r}$ viewed as a rotation about a fixed axes $\theta=\theta \mathbf{e}$ according to Euler's theorem; on the right a top view along the $\theta$ axis is shown.

### 9.6. LARGE ROTATIONS

The treatment of large rotations in three dimensions for shells and beams is described in the following. This topic has been extensively explored in the literature on large displacement finite element methods and multi-body dynamics, Shabana (). Large rotations are usually treated by Euler angles in classical dynamics texts. However, Euler angles are nonunique for certain orientations and lead to awkward equations of motion. Therefore alternative techniques which lead to cleaner equations are usually employed. In addition, in 5 degree-of-freedom shells formulations, the rotation should be treated as two dependent variables. These factors are discussed in the following.

Euler's Theorem and Exponential Map. The fundamental concept in the treatment of large rotations is the theorem of Euler. This theorem states that in any rigid body rotation there exists a line which remains fixed, and the body rotates about this line This formula enables the development of general formulas for the rotation matrix: some special cases which will be described here are the Rodrigues formulas and the Hughes-Winget update. other techniques are quaternion, Cardona and Geradin().

The fundamental equation which evolves from Euler's theorem is the rotation formula which relates the components of a vector $r^{r}$ in a rigid body which is rotated about the axis $\theta$. The vector after the rotation is denoted by $\mathbf{r}^{\prime}$ as shown in Fig. 9.?. The objective then is to obtain a rotation matrix $\mathbf{R}$ so that

$$
\begin{equation*}
\mathbf{r}^{\prime}=\mathbf{R r} \tag{9.6.1}
\end{equation*}
$$

The nomenclature to be used is shown in Fig. ?, where the line segment about which the body rotates is denoted by the unit vector $\mathbf{e}$. We will first derive the formula

$$
\begin{equation*}
\mathbf{r}^{\prime}=\mathbf{r}+\sin \theta \mathbf{e} \times \mathbf{r}+(1-\cos \theta) \mathbf{e} \times(\mathbf{e} \times \mathbf{r}) \tag{9.6.2}
\end{equation*}
$$

The schematic on the right of Fig. ? shows the body as viewed along the $\mathbf{e}$ axis. It can be seen from this schematic that

$$
\begin{equation*}
\mathbf{r}^{\prime}=\mathbf{r}+\mathbf{r}_{P Q}=\mathbf{r}+\alpha \sin \theta \mathbf{e}_{2}+\alpha(1-\cos \theta) \mathbf{e}_{3} \tag{9.6.3}
\end{equation*}
$$

where $\alpha=r \sin \phi$. From the definition of the cross product it follows that

$$
\begin{equation*}
\alpha \mathbf{e}_{2}=r \sin \phi \mathbf{e}_{2}=\mathbf{e} \times \mathbf{r}, \quad \alpha \mathbf{e}_{3}=r \sin \phi \mathbf{e}_{3}=\mathbf{e} \times(\mathbf{e} \times \mathbf{r}) \tag{9.6.4}
\end{equation*}
$$

Substituting the above into () yields Eq. ().
We now develop a matrix so that (??) can be written in the form of a matrix multiplication. For this purpose, weuse the same scheme as in (3.2.35) to define a skewsymmetric tensor $\Omega(\theta)$ so that

$$
\begin{equation*}
\Omega(\theta) \mathbf{r}=\theta \times \mathbf{r} \tag{9.6.5}
\end{equation*}
$$

In other words, we define a matrix $\Omega(\theta)$ that has the same effect on $\mathbf{r}$ as the cross product with $\theta$. Recall from (3.2.35) that the skew-symmetric tensor equivalent to a cross product with a vector can be obtained by defining $\Omega(\theta)$ by $\Omega_{i j}(\theta)=e_{i j k} \theta_{k}$ where $e_{i j k}$ is the alternator tensor. From this definition of the $\Omega(\theta)$ matrix it follows that

$$
\begin{equation*}
\Omega(\mathbf{e}) \mathbf{r}=\mathbf{e} \times \mathbf{r}, \quad \Omega^{2}(\mathbf{e}) \mathbf{r}=\Omega(\mathbf{e}) \Omega(\mathbf{e}) \mathbf{r}=\mathbf{e} \times(\mathbf{e} \times \mathbf{r}) \tag{9.6.6}
\end{equation*}
$$

Comparing the above terms with (), it can be seen that (??) can be written as

$$
\begin{equation*}
\mathbf{r}^{\prime}=\mathbf{r}+\sin \theta \Omega(\mathbf{e}) \mathbf{r}+(1-\cos \theta) \Omega^{2}(\mathbf{e}) \mathbf{r} \tag{9.6.7}
\end{equation*}
$$

so that comparison with () shows that

$$
\begin{equation*}
\mathbf{R}=\mathbf{I}+\sin \theta \Omega(\mathbf{e})+(1-\cos \theta) \Omega^{2}(\mathbf{e}) \tag{9.6.8}
\end{equation*}
$$

In writing the rotation matrix, it is often useful to define a vector $\theta$ along the axis of rotation $\mathbf{e}$ with length $\theta$, the angle of rotation. The vector $\theta$ is sometimes called a pseudovector because sequential rotations cannot be added as vectors to obtain the final
rotation, i.e. if the pseudovector $\theta_{12}$ corresponds to the rotation $\theta_{1}$ followed by the rotation $\theta_{2}$ then $\theta_{12} \neq \theta_{21}$. This property of rotations is often illustrated in introductory physics texts by rotating an object such as a book 90 degrees about the x -axis followed by a 90 degree rotation about the $y$-axis and comparing this with a 90 degree rotation about the $y$-axis followed by a 90 degree rotation about the x -axis.

An important way to describe rotation is the exponential map, which gives the rotation matrix $\mathbf{R}$ by

$$
\begin{equation*}
\mathbf{R}=\exp (\Omega(\theta))=\sum_{n} \frac{\Omega^{n}(\theta)}{n!}=\mathbf{I}+\Omega(\theta)+\frac{\Omega^{2}(\theta)}{2}+\frac{\Omega^{3}(\theta)}{6}+\ldots \tag{9.6.9}
\end{equation*}
$$

This form of the rotation matrix can be used to obtain accurate approximation to the rotation matrix for small rotations. To develop the exponential map from (9.6.8) we note that the matrix $\Omega(\theta)$ satisfies the following recurrence relation

$$
\begin{equation*}
\Omega^{n+1}(\theta)=-\theta \Omega^{n}(\theta) \tag{9.6.10}
\end{equation*}
$$

This relationship can be obtained easily by using the interpretation of $\Omega(\theta)$ as a matrix which replicates the cross-product as given in Eq. (9.6.5) and that it scales with $\theta$. The trigonometric functions $\sin \theta$ and $\cos \theta$ can be expanded in Taylor's series

$$
\begin{equation*}
\sin \theta=\theta-\frac{\theta^{3}}{3!}+\frac{\theta^{5}}{5!}-\ldots, \quad \cos \theta=1-\frac{\theta^{2}}{2!}+\frac{\theta^{4}}{4!}-\ldots \tag{9.6.11}
\end{equation*}
$$

yielding (9.6.9).

### 9.7. SHEAR AND MEMBRANE LOCKING

Among the most troublesome characteristics of shell elements are shear and membrane locking. Shear locking results from the spurious appearance of transverse shear in deformation modes that should be free of transverse shear. More precisely, it emanates from the inability of many elements to represent deformation modes in which the transverse shear should vanish. Since the shear stiffness is often significantly greater than the bending stiffness, the spurious shear absorbs a large part of the energy imparted by the external forces and the predicted deflections and strains are much too small, hence the name shear locking.

The observed behavior of thin beams and shells indicates that the normals to the midline remain straight and normal, and that hence the transverse shears vanish. This behavior can be viewed as a constraint on the motion of the continuum. While the normality constraint is not exactly enforced in the shear-beam or CB shell theories, the normality constant always appears as a penalty term in the energy. The penalty factor increases as the thickness decreases, see Example (??), so as the thickness decreases shear locking becomes more prominent. Shear locking does not appear in $\mathrm{C}^{1}$ elements, since the motion in $C^{1}$ elements is such that the normals remain normal. In $C^{0}$ (and $C B$ structural) elements, the normal can rotate relative to the midline, so spurious transverse shear and locking can appear.

Membrane locking results from the inability of shell finite elements to represent inextensional modes of deformation. A shell can bend without stretching: take a piece of paper and see how easily you can bend it. However stretching a piece of paper is almost impossible. Shells behave similarly: their bending stiffness is small but their membrane stiffness is large. So when the element cannot bend without stretching, the energy is incorrectly shifted to membrane energy, resulting in underprediction of displacements and strains. Membrane locking is particularly important in simulation of buckling since many buckling modes are completely or nearly inextensible.

The situation for shear and membrane locking is similar to the volumetric locking described in Chapter 8: a finite element approximation to motion cannot represent a motion in a constrained medium that satisfies the constraint, where the constraint is much stiffer than the stiffness experienced by the correct motion. In the case of volumetric locking, the constraint is incompressibility, while the in the case of shear and membrane locking the strains are the normality constraint of Kirchhoff-Love behavior and the inextensibility constraint. This is summarized in Table 9.??. It should be noted that the Kirchhoff-Love behavior of thin shells, and the counterpart in Euler-Bernoulli beams, is not an exact constraint. For thicker shells and beams, some transverse shear is expected, but just as elements that lock exhibit poor performance for nearly incompressible materials, shell elements which lock in shear perform poorly for thick shells where transverse shear is expected.

Table 9
Analogy of Locking Phenomena

| Constraint | Shortcoming of finite <br> element motion | Locking type |
| :--- | :--- | :--- |
| incompressibility <br> isochoric motion <br> $J=$ constant, $v_{i, i}=0$ | volumetric strain appears in <br> element | volumetric locking |
| Kirchhoff-Love constraint <br> $\hat{D}_{x z}=\hat{D}_{y z}=0$ | transverse shear strain <br> appears in pure bending | shear locking |
| Inextensibility constraint | membrane strain appears in <br> inextensional mode | membrane locking |

Shear Locking. This description of shear and membrane locking closely follows Stolarski, Belytschko and Lee ( ). To illustrate shear locking, we consider the two-node beam element described in Example 9.1; for simplicity, consider the element being along the $x$-axis. Since shear and membrane locking occur in linear response of beams and shells, our examination will be made in the context of linear theory. The transverse shear strain is given by

$$
\begin{equation*}
2 \varepsilon_{x y}=\frac{1}{\ell}\left(u_{x 2}^{M}-u_{x 1}^{M}\right)-\theta_{1}(1-\xi)-\theta_{2} \xi \tag{9.7.1}
\end{equation*}
$$

We now consider the element in a state of pure bending, where the moment $m(x)$ is constant. From the equilibrium equation, Eq. (??), the shear $s(x)$ should vanish when the
moment is constant. We now consider a specific deformation mode of the element where the moment is constant:

$$
\begin{equation*}
u_{x 1}=u_{x 2}=0, \theta_{1}=-\theta_{2}=\alpha . \tag{9.7.2}
\end{equation*}
$$

It is easy to verify the bending moment is constant for this element, and anyway the deformation can be seen to be a pure bending mode. For these nodal displacements, Eq. (9.7.1) gives

$$
\begin{equation*}
2 \varepsilon_{x y}=\alpha(2 \xi-1) \tag{9.7.3}
\end{equation*}
$$

Thus the transverse shear strain, and hence the transverse shear stress, are nonzero is most of the element, which contradicts the expected behavior that the transverse shear vanish in when the moment is constant.

To explain why this parasitic transverse shear has such a large effect, the energies associated with the various strains are examined for a linear, elastic beam of unit depth with a rectangular cross-section. The bending energy is the energy associated with the linear portion of the axial strains, which is given by

$$
\begin{equation*}
W_{b e n d}=\frac{E}{2} \int_{\Omega} y^{2} \theta_{, x}^{2} d \Omega=\frac{E h^{3}}{24} \int_{0}^{\ell} \theta_{, x}^{2} d x=\frac{E h^{3}}{24 \ell}\left(\theta_{2}-\theta_{1}\right)^{2}=\frac{E h^{3} \alpha^{2}}{6 \ell} \tag{9.7.4a}
\end{equation*}
$$

where the rotations associated with the bending mode (9.7.2) have been used in the last expression.

The shear energy for the beam is given by

$$
\begin{equation*}
W_{\text {shear }}=\frac{E}{(1+v)} \int_{\Omega} \varepsilon_{x y}^{2} d \Omega=\frac{E h}{(1+v)} \int_{0}^{\ell}\left(\theta-u_{z, x}\right)^{2} d x=\frac{E h \ell \alpha^{2}}{3(1+v)} \tag{9.7.4b}
\end{equation*}
$$

The ratio of these two energies is given approximately by

$$
\frac{W_{\text {shear }}}{W_{\text {bend }}} \approx\left(\frac{\ell}{h}\right)^{2}
$$

Thus for a thin element with the length $\ell$ greater than the thickness $h$ the shear energy is greater than the bending energy. Since the shear energy should vanish in pure bending, the effect of this parasitic shear energy is a significant underprediction of the total displacement. As the length of the element decreases due to mesh refinement, the ratio of shear to bending energy in each element decreases, but the convergence tends to be very slow. However, in contrast the volumetric locking, where often no convergence is observed with refinement, elements that lock in shear converge to the correct solution, but very slowly.

Equation (9.7.3) immediately suggests why underintegration can alleviate shear locking in this element: note that the transverse shear vanishes at $\xi=\frac{1}{2}$, which
corresponds to the quadrature point in one-point quadrature. Thus, the spurious transverse shear is eliminated by underintegration of the shear-related terms.

Membrane Locking. In the following, we will use linear strain displacement equations, which are only valid for small strains and rotations to explain shear and membrane locking. To illustrate membrane locking we will use the Marguerre shallow beam equation. The Marguerre equations are

$$
\begin{align*}
& \varepsilon_{x}=u_{x, x}^{M}+w_{, x}^{0} u_{z, x}-y \theta_{, x}  \tag{9.7.5a}\\
& 2 \varepsilon_{x y}=u_{z, x}-\theta \tag{9.7.5b}
\end{align*}
$$

It should be stressed that while these kinematic relations are different from the CB beam equations, they in fact closely approximate the CB equations for shallow beams, i.e. when $w^{0}(x)$ is small. The mechanical behavior predicted by the various theories for a thin beam is almost identical if the assumptions of the theories are met. For shallow beams, Marguerre theory gives very accurate results.

Consider a three-node beam element. In an inextensional mode, the membrane strain $\varepsilon_{x}$ must vanish, so by integrating the expression for $u_{x, x}^{M}$ in (9.7.5a) for $y=0$ it follows that

$$
\begin{equation*}
u_{x 3}^{M}-u_{x 1}^{M}=-\int_{0}^{\ell} w_{, x}^{0} w_{, x} d x \tag{9.7.6}
\end{equation*}
$$

Consider a beam in a pure bending mode so $\theta_{1}=-\theta_{3}=\alpha$. In the absence of transverse shear it follows from Eq. (9.4.5b) that

$$
\begin{equation*}
u_{z 2}=\int_{0}^{\ell / 2} \theta(x) d x=\frac{\alpha \ell^{2}}{4} \tag{9.7.7}
\end{equation*}
$$

Consider a beam in an initially symmetric configuration, so $\theta_{1}^{0}=\theta_{3}^{0}=\alpha_{0}, \theta_{2}^{0}=0$. Then Eq. (9.7.6) is satisfied if $u_{x 1}=-u_{x 3}=\frac{\alpha_{0} \alpha \ell}{6}, u_{x 2}=0$. Evaluation of the membrane strain via Eqs. (??) and (9.7.5a) then gives

$$
\begin{equation*}
\varepsilon_{x}=\alpha \alpha_{0}\left(\frac{1}{3}-\xi^{2}\right) \tag{9.7.8}
\end{equation*}
$$

Thus, in this particular inextensional mode of deformation, the extensional strain does not vanish throughout the beam. If an element is developed with a quadrature scheme which includes quadrature points where the extensional strain does not vanish, the element will exhibit membrane locking.

The possibility of membrane locking in the three-node curved beam can also be determined by examining the orders of the displacement fields. The variables
$u_{x}, u_{y}$, and $w^{0}$ are quadratic, and the quadratic fields are actuated in a pure bending mode. Since $u_{x, x}$ is linear, the membrane strain Eq. (9.7.5a) cannot vanish uniformly throughout the element in a pure bending mode if $W^{0}$ is nonzero. Thus membrane locking can be said to originate from the inability of the finite element interpolant to represent inextensional motions. Shear locking can be explained similarly as the inability of finite element interpolants to represent pure bending modes.

From the preceding, an obvious remedy to membrane and shear locking would be to match the order of the interpolants of different components of the motion. For example, is a cubic field $u_{x}$ would improve the representation of an inextensional mode for quadratic $u_{y}$. However, it is difficult to accomplish this within the framework of CB elements based on isoparametric elements without disturbing the element's capacity to represent rigid body motion, which is crucial for convergence.

If the element is rectilinear, $w^{0}$ vanishes and membrane locking will not occur because bending will not generate membrane strains, see Eq. (9.7.5a). Membrane locking does not occur in flat shell elements or straight beam elements. Thus, the two-node beam never exhibits membrane locking and the four-node quadrilateral shell only manifests membrane locking in warped configurations.

Shear locking in the three-node beam is less obvious than for the two-node beam. The shear strain in this element is given by

$$
\begin{align*}
2 \varepsilon_{x y}=u_{z, \theta}-\theta= & \frac{1}{\ell}\left[(2 \xi-1) u_{z 1}-4 \xi u_{z 2}+(2 \xi+1) u_{z 3}\right]  \tag{9.7.9}\\
& -\frac{1}{2}\left(\xi^{2}-\xi\right) \theta_{1}-\left(\xi^{2}-1\right) \theta_{2}-\frac{1}{2}\left(\xi^{2}-\xi\right) \theta_{3}
\end{align*}
$$

Consider a state of pure bending, $\theta_{1}=-\theta_{3}=\alpha, \theta_{2}=0, u_{z 1}=u_{z 3}$ and $u_{z 2}=\frac{\alpha \ell}{4}$. Using these nodal displacements in Eq. (9.7.9) gives a vanishing transverse shear throughout the element. However, consider nodal displacements for another bending mode in which the transverse shear should vanish, $u_{z}^{M}=\alpha \xi^{3}, \theta=u_{z, x}=\frac{6 \alpha \xi^{2}}{\ell}$. According to Eq. (9.7.9)

$$
\begin{equation*}
2 \varepsilon_{x y}=\frac{\alpha}{\ell}\left(1-3 \xi^{2}\right) \tag{9.7.10}
\end{equation*}
$$

so the finite element approximation gives nonzero shear.
Remarkably, the shear in Eq. (9.7.10) and the membrane strains in Eq. (9.7.8) vanish at the points $\xi= \pm 1 \sqrt{3}$, which correspond to the Gauss quadrature points for twopoint quadrature. These are often called the Barlow points because Barlow [53] first pointed out that at these points of an eight-node isoparametric element, if the nodal displacements are set by a cubic field, the stresses obtained via the strain-displacement equations and stress-strain laws also correspond to those obtained from a cubic displacement field. He concluded that "if the element is used to represent a general cubic displacement field, the stresses at the $2 \times 2$ Gauss points will have the same degree of
accuracy as the nodal displacements." While it is not clear whether the Barlow hypothesis applies directly to elements such as the nine-node Lagrange shell element, the serendipitous features of the Gauss quadrature points in quadratic elements are undeniable.

Although this model for membrane locking is based on the shallow shell equations, it correctly predicts the performance of elements developed by other shell theories or degenerated continuum elements. The mechanical behavior of elements is almost independent of the underlying shell theory as long as the element is shallow. Moreover, as meshes are refined, elements increasingly conform to the shallow shell hypothesis. However, the extension of these concepts and analyses to general shell elements is quite difficult, particularly when the element is not rectangular. For nonrectangular elements, the development of reduced quadrature schemes or assumed strain fields for shells which avoid both shear and membrane locking has been a challenging task which is not fully resolved for elements of quadratic order or higher.

The fact that the shear strain vanishes at the Barlow points explains the success of reduced integration as introduced by Zienkiewicz et al. [54]. When the shear strain is only sampled at the Barlow points in integrating the shear stiffness, it will not sample the spurious shear which occurs along the remainder of the beam. Similarly, the shear strain in the two-node element, (3.1.7) vanishes at $\xi=0$. Therefore, if the shear is only sampled at this point, shear locking is avoided (see [55]).

The alleviation or complete elimination of these two locking phenomena has been a central thrust of plate and shell element research. This has not proven an easy task, particularly when combined with the goal of not permitting any spurious singular zero energy modes in the element.

### 9.8 ASSUMED STRAIN ELEMENTS

To circumvent the difficulties of shear and membrane locking, it is necessary to develop assumed shear and membrane strain fields which avoid spurious (or parasitic) shear and membrane strains. Shear and membrane locking can also be avoided by selective-reduced integration, but selective-reduced integration is not as successful in shells as in continua. For example, in the quadrilateral four-node shell element described in Hughes (?? p ?), the element with selective-reduced integration still possesses a spurious singular mode, the w-hourglass mode, see Belytschko and Tsay (??). Thus while selective-reduced integration provides robust elements for continua, it is not as successful for shells.

The assumed strain methods are based on mixed variational principles, such as the Hu-Washizu and the Simo-Hughes B-bar simplification. When the CB shell methodology is employed, the mixed principles can be employed in the same form as given for continua; for those who have not yet appreciated CB shell theory, one element in their attractiveness is that it eliminates for reformulating the many ingredients of continuous finite elements for shells.

The Hu-Washizu weak form is then given by

$$
\begin{equation*}
\delta \pi^{H W}(\mathbf{u}, \bar{\sigma}, \overline{\mathbf{D}})=\int_{\Omega}\left[\delta \mathbf{D}: \bar{\sigma}-\delta \bar{\sigma}\left(\bar{V}_{s} \mathbf{v}-\overline{\mathbf{D}}\right)\right] d \Omega-\delta W^{e x t} \tag{9.8.1}
\end{equation*}
$$

where we note that $\hat{\sigma}_{z z}=0$ because of the plane-stress assumption.
The essence of the assumed strain approach is then to design transverse shear fields and membrane strain fields so that shear and membrane locking are mitigated. This is done by eliminating the strains which are parasitic: transverse shear strains in bending and membrane strains in inextensional bending. Furthermore, these strain fields must be designed so that the correct rank of the stiffness matrix is retained to avoid spurious singular modes. In the following, we concentrate on the B-bar form of the mixed field implementation, so once the strain fields have been designed, the internal nodal forces at the slave nodes are given by

$$
\begin{equation*}
\{f\}^{i n t}=\int \overline{\mathbf{B}}^{T}\{\sigma\} \tag{9.8.2}
\end{equation*}
$$

9.8.2. Assumed Strain Four-Node Quadrilateral. The shape functions and motion of the four-node quadrilateral shell element based on the 8 -node hexahedron were given in Example ??. The objective here is to construct the shear and membrane strain fields so that locking is avoided.

The construction of the transverse shear field for the four-node quadrilateral is motivated by Eq. (9.7.3), the transverse shear distribution for a beam in bending. We examine this first for a rectangular shell element. A rectangular shell element behaves similarly to a beam, so when a bending moment is applied as shown in Figure 9.?, the transverse shear $\sigma_{x z}$ should vanish. When the material is isotropic, this can be met if $\bar{D}_{x z}$ vanishes, and this can be effected by making it constant in the $x$-direction. So the assumed shear is taken to be

$$
\bar{D}_{x z}=\alpha_{1}
$$



Figure 9.?. Rectangular element under pure bending showning the transverse shear which is activated, if not suppressed, by assumed strain methods.

However, a constant transverse shear leads to a rank deficiency in the element. To restore stability, a linear dependence on $y$ is added: this extra field has no effect on the behavior on bending due to the moment $m_{y y}$, so the unlocking is not disturbed. So

$$
\bar{D}_{x z}=\alpha_{1}+\alpha_{2} y
$$

In the application of the Hu-Washizu weak form, the parameters would be found by the discrete compatibility equations. However, this complicates the computation of the element. Instead, the above shear fields are interpolated directly from values of the shear at selected points. In this case, the midpoints of the edges are chosen as interpolation points. The shear field is given by

$$
D_{x z}=\frac{1}{2}\left(D_{x z}\left(\xi_{a}, t\right)+D_{x z}\left(\xi_{b}, t\right)\right)+\frac{1}{2} D_{x z}\left(\xi_{a}, t\right)(1-\eta)+\frac{1}{2} D_{x z}\left(\xi_{b}, t\right)(1+\eta)
$$

where

$$
\begin{array}{ll}
\xi_{a}=\left(\frac{1}{2},-1,0\right) & \xi_{b}=\left(\frac{1}{2}, 1,0\right) \\
\xi_{c}=\left(-1, \frac{1}{2}, 0\right) & \xi_{d}=\left(1, \frac{1}{2}, 0\right)
\end{array}
$$

The points are shown in Figure 9.?. We have used $\eta$ instead of $y$ since $y=2 b \eta$ in this element. By similar arguments, see Figure 9.?, the transverse shear $\bar{D}_{y z}$ is interpolated by

$$
\bar{D}_{y z}=\frac{1}{2}\left(D_{y z}\left(\xi_{c}, t\right)+D_{y z}\left(\xi_{d}, t\right)\right)+\frac{1}{2} D_{y z}\left(\xi_{c}, t\right)(1-\xi)+\frac{1}{2} D_{y z}\left(\xi_{d}, t\right)(1+\xi)
$$

To extend this technique to quadrilaterals, it was noted that $D_{\xi z}$ vanishes when the moment $m_{\eta \eta}$ is constant, and $D_{\eta z}$ vanishes when $m_{\xi \xi}$ is constant.

The assumed strain field given here was first constructed on the basis of physical arguments by MacNeal () an identical field was used by Hughes and Tezduyar (??); the referential interpolation was given by Wempner and Talislides (??). Dvorkin and Bathe (??) constructed the field given in the previous references on the basis of interpolation.

The basic idea is to assume the transverse shears so that under a constant movement about the $\eta$-axis, the resulting transverse shear, $D_{Z \xi}$ is constant, with a similar argument for $D_{Z \eta}$. The resulting shear fields are

$$
\begin{align*}
& \bar{D}_{z \xi}(\xi, \eta, \zeta, t)=\alpha_{1}+\alpha_{2} \eta  \tag{9.8.5a}\\
& \bar{D}_{z \eta}(\xi, \eta, t)=\beta_{1}+\beta_{2} \xi \tag{9.8.5b}
\end{align*}
$$

where $\alpha_{i}$ and $\beta_{i}$ are arbitrary parameter. As can be seen from the above, the shear $D_{Z \xi}$ has no variation in the $\xi$ direction, so when a moment is applied about the $\xi$ axis, the
relevant shear is constant. However, a $\eta$ dependence has been added to stabilize the elment, i.e. to correct the rank. Analogous reasoning is used for the construction of the shear field $\bar{D}_{z \eta}$.

To avoid the Hu-Washizu weak form, the midpoints of the edges are chosen as interpolation. The shear fields are given by

$$
\begin{aligned}
& \bar{D}_{z \xi}(\xi, \eta, \zeta, t)=\frac{1}{2} D_{z \xi}(0,-1,0, t)(1-\eta)+\frac{1}{2} D_{z \xi}(0,1,0, t)(1+\eta) \\
& \bar{D}_{z \eta}(\xi, \eta, \zeta, t)=\frac{1}{2} D_{\hat{z} \eta}(-1,0,0, t)(1-\xi)+\frac{1}{2} D_{z \xi}(1,0,0, t)(1+\eta)
\end{aligned}
$$

where $D_{z \xi}(\xi, \bar{\Delta}, t)$ and $D_{z \eta}\left(\xi_{a}, \eta_{\mathrm{a}}, 0, t\right)$ are the velocity strains computed at the midpoints of the edges from the velocity field.

Assumed strain fields for the nine-node shell that avoid membrane and shear locking have been given by Huang and Hinton (1986) and Bucalem and Bathe (1993). We just briefly describe the latter.

In this scheme, the velocity strains $D_{\xi \xi}$ and $D_{\xi \xi}$ are interpolated by using the corresponding velocity strains computed at the six points shown in Figure ?? and a linearquadratic isoparametric field, so

$$
\begin{aligned}
& \bar{D}_{\xi \xi}=D_{\xi \xi}\left(\xi_{I}, \eta_{J}, 0, t\right) N_{I J}(\xi, \eta) \\
& \bar{D}_{\zeta \zeta}=D_{\xi \zeta}\left(\xi_{I}, \eta_{J}, 0, t\right) N_{I J}(\xi, \eta)
\end{aligned}
$$

where $N_{I J}(\xi, \eta)$ are shape functions formed by the product of Lagrange interpolants linear in $\xi$ and quadratic in $\eta$ so that

$$
N_{I J}\left(\xi_{K}, \eta_{L}\right)=\delta_{I K} \delta_{J L}
$$

Note that the curvilinear components are interpolated, including the replacement of $D_{\xi Z}$ by $D_{\xi \xi}$; it is not clear whether the latter offers any advantage. The interpolation of $D_{\xi \xi}$ is convenient because it relates the component interpolated to the parent element coordinates, so that the stiffness of the element is independent of the orientation of the element. Although no motivation is given for the selection of the interpolation points in Bathe (1998), the beam example in the previous Section sheds some light on it: at the Gauss quadrature points $\pm 3^{-1 / 2}$, the transverse shear vanishes in bending and the membrane straine vanished in inextensional bending. Thus the element should not exhibit spurious transverse shears or membrane strains. The higher order interpolation in the $\eta$ direction provides stability.

The velocity strains $\bar{D}_{\eta \eta}$ and $\bar{D}_{\eta \zeta}$ are interpolated with the rotated image of (??). The shear component $\bar{D}_{\xi \eta}$ is interpolated with another set of points shown in Figure 9.?.

### 9.9. ONE-POINT QUADRATURE ELEMENTS.

In explicit software and large scale implicit software, the most widely used shell elements are four-node quadrilaterals with one-point quadrature. Here the one-point quadrature refers to the number of quadrature points in the reference plane: actually, anywhere from three to thirty or more quadrature are used through the thickness, depending on the complexity of the nonlinear material response. Therefore, we often refer to one stack of quadrature points. The number of quadrature points is actually one only for resultant stress theories. For CB elements the motion of the element is based on eight-node hexahedron continuum element, although the description of the motion is often simplified to the four-node quadrilateral shape functions on the reference surface.

These elements are the most commonly used in large-scale analysis because they work well with diagonal mass matrices and are extremely robust. Higher order elements, such as those based on quadratic isoparametrics, converge more rapidly to smooth solutions. However, most large-scale analyses involve nonsmooth problems, with elastoplasticity, contact-impact, etc., so the greater approximation power of higher order elements is not realizable in these problems.

Since only one stack of quadrature points are used, the element is, unless hourglass control is added, rank-deficient and unstable. Therefore, hourglass control is required to stabilize the element. In the following, the various forms of hourglass control are also described.

We will first summarize the elements which have been most frequently used in software. We then describe two of these elements in more detail, drawing on the material which precedes this to abbreviate the description.

The elements used most frequently are listed in Table X, along with some of the most prominent features and drawbacks. The earliest is the Belytschko-Tsay (BT) element, which is based on Belytschko and Tsay (1983) and Belytschko, Liu, and Tsay (1984). It is constructed by combining a flat, four-node element with a plane quadrilateral four-node membrane. As indicated in Table X, it dos not respond correctly when its configuration is warped (this shortcoming manifests itself primarily when one or two lines of elements are used to model twisted beams, as described later). However, the element is very robust and fast. Whereas most of the other elements often fail when subjected to severe distortions, the BT element seldom aborts a computation. This is highly valued in industrial settings.

The Hughes-Liu (HL) element, partially described in Hughes and Liu (1981), is CB shell element. In explicit codes, it is used with a single stack of quadrature points, so it also requires hourglass control and the techniques developed in Belytschko, Liu and Tsay (1984) are used. It is significantly slower than the BT element.

The BWC element corrects the twist, i.e., the warped configuration defect in the BT element. Otherwise, it is quite similar. In the BL element, the so-called physical hourglass control described in Chapter 8 is implemented. This hourglass control is based on a multifield variational principle, so it is theoretically possible to exactly reproduce the behavior of a fully integrated element. However, in practice this is possible only for
elastic response, since the homogeneity of the strain and stress state are crucial in obtaining closed form expressions for the physical hourglass control. Nevertheless, this form of hourglass control provides a substantial advantage; it can be increased to moderately large values without inducing locking; whereas in the BT element high values of the hourglass control parameters result in locking.

Both the BL element and the fully integrated element are afflicted with another shortcoming. In problems with large distortions, these elements fail suddenly and dramatically, aborting the simulation. So the advantage of single quadrature point elements does not reside only in their superior speed, in addition, they tend to be more robust.

The YASE element (yet another shell element) incorporates the Pian-Sumihara (1984) membrane field for improved membrane response in beam bending, i.e., for improved flexural performance, as described in Section 8.?. Otherwise, it is identical to the BT element.

The BT, BWC, and BL elements are based on a discrete Mindlin-Reissner theory which is not continuum-based. "Discrete" refers to the fact that the assumption is only applied to the motion at the quadrature point. The motion is constrained by requiring the current normal to remain straight. This can be viewed as another modification of the Mindlin-Reissner assumption in its extension to large deformations; rather than requiring the initial normal to remain straight, the current normal is required to remain straight. The effectiveness of this assumption as compared to the assumption in Section 9.8 can be judged only by comparison to experiment. The velocity in the element is given by... A corotational coordinate system is used. Although in the original papers the corotational coordinate system was aligned with $\hat{e}_{x}$ along $\mathbf{x}, \xi$, this can lead to difficulties, so the technique descried in Section 8.?. is used.

The current configuration of the element is shown in Figure 9.?. As can be seen, $\hat{e}_{z}$ is always normal to the reference surface at the location of the quadrature point stack. The velocity field is given by Eq. (9.8.7) with the thickness rate dropped:

$$
\begin{equation*}
\mathbf{v}(\xi, t)=\mathbf{v}^{M}(\xi, \eta, t)+\bar{\xi}(\omega(\xi, \eta, t) \times \tilde{\mathbf{p}}(\xi, \eta, t)) \tag{9.9.1}
\end{equation*}
$$

where a curlicue is superimposed on the nodal director $\tilde{\mathbf{p}}_{I}$ to indicate that it may differs from the director as defined in Section 9.8. The finite element approximation to the motion is

$$
\begin{equation*}
\mathbf{v}(\xi, t)=\sum_{I=1}^{4}\left(v_{I}(t)+\bar{\zeta} \omega_{I}(t) \times \mathbf{p}_{I}\right) N_{I}(\xi, \eta) \tag{9.9.2}
\end{equation*}
$$

Converting the cross-product to a scalar product, the above can be written

$$
\begin{equation*}
\mathbf{v}(\xi, t)=\sum_{I=1}^{4}\left(v_{I}(t)+\bar{\zeta} \Omega \mathbf{p}_{I}\right) N_{I}(\xi, \eta) \tag{9.9.2c}
\end{equation*}
$$

where $N_{I}$ are the four-node isoparametric shape functions.
The rate-of-deformation tensor in corotational form is

$$
\begin{equation*}
\{\mathbf{D}\}^{T}=\left[D_{\hat{x} \hat{x}}, D_{\hat{y} \hat{y}}, 2 D_{\hat{x} \hat{y}}, 2 D_{\hat{x} \hat{z}}, 2 D_{\hat{y} \hat{z}}\right] \tag{9.9.2b}
\end{equation*}
$$

where $D_{\hat{z} \hat{z}}$ is omitted since it does not contribute to the power because of the plane stress condition. The components are evaluated by Eq. (3.2.39).

The rate-of-deformation is evaluated by using a linear expansion of the Jacobian $J$ in corotational coordinate system $[\hat{x}, \hat{y}]$. It has been found that a linear expansion captures the major effects, such as twist, for thin shells. To make this expansion, the shape functions are considered in three-dimensional form. The linear expansion is of the shape function derivatives is then

$$
\left\{\begin{array}{l}
N_{I, \hat{x}}  \tag{9.9.3}\\
N_{I, \hat{y}}
\end{array}\right\}=\left\{\begin{array}{l}
N_{I, \hat{x}} \\
N_{I, \hat{y}}
\end{array}\right\}+\bar{\xi}\left\{\begin{array}{l}
b_{x I}^{c} \\
b_{y I}^{c}
\end{array}\right\}
$$

where

$$
\left\{\begin{array}{c}
b_{x I}^{c}  \tag{9.9.4}\\
b_{y I}^{c}
\end{array}\right\}=\frac{1}{J}\left[\begin{array}{cc}
p_{\hat{y}, \eta} & -p_{\hat{y}, \xi} \\
-p_{\hat{x}, \eta} & p_{\hat{x}, \xi}
\end{array}\right]\left\{\begin{array}{l}
N_{I, \xi} \\
N_{I, \eta}
\end{array}\right\}
$$

The director $\mathbf{p}$ is taken to be the normal in the current configuration (the director changes with time and is not the tangent to a material fiber). Setting $\mathbf{p}$ to the normal to the reference surface

$$
\mathbf{p}=\frac{1}{p^{*}}\left\{\begin{array}{c}
-\hat{z}_{, \hat{x}}^{M}  \tag{9.9.5}\\
-\hat{z}_{, \hat{y}}^{M} \\
1
\end{array}\right\}=-\frac{1}{p^{*}} \sum_{I=1}^{4} \hat{z}_{I}\left\{\begin{array}{c}
b_{x I}+(\xi \eta)_{, \hat{x}} \gamma_{I} \\
b_{y I}+(\xi \eta),{ }_{\hat{y}} \gamma_{I} \\
1
\end{array}\right\}
$$

where

$$
\begin{equation*}
p^{*}=\left(1+\hat{z}_{, \hat{x}}^{2}+\hat{z}_{, \hat{y}}^{2}\right)^{1 / 2} \tag{9.9.6}
\end{equation*}
$$

and $\gamma_{I}$ is the consistent hourglass operator given in Section 8.?. At the origin $(\xi \eta)_{, \hat{x}}=(\xi \eta)_{, \hat{y}}=0$, because

$$
\begin{equation*}
(\xi \eta)_{, \hat{x}}=\xi \eta_{, \hat{x}}=\xi_{, \hat{x}} \eta=0 \tag{9.9.7}
\end{equation*}
$$

The director $\mathbf{p}$ is constructed normal to the $\hat{x}-\hat{y}$ plane at the origin, so from and Eq. (9.9.7), it follows that

$$
\begin{equation*}
\sum_{I=1}^{4} b_{x I} \hat{z}_{I}=\sum_{I=1}^{4} b_{y I} \hat{z}_{I}=0 \tag{9.9.8}
\end{equation*}
$$

Therefore, $p^{*}=1$ at the origin of the reference plane, i.e. at the quadrature point.
Taking the derivatives of $p_{\hat{x}}$ and $p_{\hat{y}}$ with respect to $\xi$ and $\eta$ (and neglecting the terms related to $p_{, \xi}^{*}$ and $p_{, \eta}^{*}$, which can be shown to be small) gives

$$
\begin{align*}
& p_{\hat{x}, \xi}=-\hat{z}_{, \hat{x} \xi}=-z_{\gamma} \eta, \hat{x}  \tag{9.9.9}\\
& p_{\hat{x}, \eta}=-\hat{z}_{, \hat{x} \eta}=-z_{\gamma} \xi_{, \hat{x}}  \tag{9.9.10}\\
& p_{\hat{y}, \xi}=-\hat{z}_{, \hat{y}}=-z_{\gamma} \eta, \hat{y}  \tag{9.9.11}\\
& p_{\hat{y}, \eta}=-\hat{z}_{, \hat{y} \eta}=-z_{\gamma} \xi_{, \hat{y}} \tag{9.9.12}
\end{align*}
$$

where

$$
\begin{equation*}
z_{\gamma}=\sum_{I=1}^{4} \gamma_{I} \hat{z}_{I} \tag{9.9.13}
\end{equation*}
$$

From Eq. (4.?.?)

$$
\left[\begin{array}{cc}
\xi_{, \hat{x}} & \xi_{, \hat{y}}  \tag{9.9.14}\\
\eta_{\hat{x}} & \eta_{, \hat{x}}
\end{array}\right]=\frac{1}{J}\left[\begin{array}{cc}
\hat{y}_{, \eta} & -\hat{x}_{, \eta} \\
-\hat{y}_{, \xi} & \hat{x}_{, \xi}
\end{array}\right]=\frac{1}{4 J}\left[\begin{array}{cc}
\hat{y}_{\eta} & -\hat{x}_{\eta} \\
-\hat{y}_{\xi} & \hat{x}_{\xi}
\end{array}\right]
$$

where

$$
\begin{equation*}
\hat{y}_{\eta}=\eta^{t} \hat{y}=\sum_{I=1}^{4} \eta_{I} \hat{y}_{I} \tag{9.9.15}
\end{equation*}
$$

It follows from Eqs. (9.9.3) and (9.9.10-14) that

$$
\begin{align*}
\left\{\begin{array}{c}
b_{x I}^{c}(\mathbf{0}) \\
b_{y I}^{c}(\mathbf{0})
\end{array}\right\} & =\frac{z_{\gamma}}{16 J^{2}}\left\{\begin{array}{l}
\xi_{I} \hat{x}_{, \eta}+\eta_{I} \hat{x}_{, \xi}(\mathbf{0}) \\
\xi_{I} \hat{y}_{, \eta}+\eta_{I} \hat{y}_{, \xi}(\mathbf{0})
\end{array}\right\}  \tag{9.9.16}\\
& =\frac{2 z_{\gamma}}{A^{2}}\left[\begin{array}{llll}
\hat{x}_{13} & \hat{x}_{42} & \hat{x}_{31} & \hat{x}_{24} \\
\hat{y}_{13} & \hat{y}_{42} & \hat{y}_{31} & \hat{y}_{24}
\end{array}\right] \tag{9.9.17}
\end{align*}
$$

Thus, the $\mathbf{b}^{c}$ column vector involves the same terms as the $\mathbf{b}$ matrix given in (???).
REMARK. Method $\hat{z}$ couples curvatures to translations only for warped elements, i.e., when the nodes are not coplanar, if which case $z_{\gamma} \neq 0$

The corotational rate-of-deformation at the quadrature point $\xi=\eta=0$ is then given by

$$
\begin{equation*}
\hat{D}_{\alpha \beta}=\hat{D}_{\alpha \beta}^{M}+\bar{\xi} \kappa_{\alpha \beta} \tag{9.9.18}
\end{equation*}
$$

where the membrane components of the rate of deformation are

$$
\begin{align*}
& \hat{D}_{x}^{M}=\frac{1}{2 A}\left(\hat{y}_{24} \hat{v}_{x 13}+\hat{y}_{13} \hat{v}_{x 42}\right)  \tag{9.9.19a}\\
& \hat{D}_{y}^{M}=\frac{1}{2 A}\left(\hat{x}_{42} \hat{v}_{y 13}+\hat{x}_{13} \hat{v}_{y 24}\right)  \tag{9.9.19b}\\
& 2 \hat{D}_{x y}^{M}=\frac{1}{2 A}\left(\hat{x}_{42} \hat{v}_{x 13}+\hat{x}_{13} \hat{v}_{x 24}+\hat{y}_{24} \hat{v}_{y 13}+\hat{y}_{31} \hat{v}_{y 24}\right) \tag{9.9.19c}
\end{align*}
$$

The curvatures are given by

$$
\begin{align*}
\kappa_{x}= & \frac{1}{2 A}\left(\hat{y}_{24} \hat{\omega}_{y 13}+\hat{y}_{31} \hat{\omega}_{y 42}\right)+\frac{2 z_{\gamma}}{A^{2}}\left(\hat{x}_{13} \hat{\omega}_{x 13}+\hat{x}_{42} \hat{v}_{x 24}\right)  \tag{9.9.20a}\\
\kappa_{y}= & -\frac{1}{2 A}\left(\hat{x}_{42} \hat{\omega}_{x 13}+\hat{x}_{13} \hat{\omega}_{x 24}\right)+\frac{2 z_{\gamma}}{A^{2}}\left(\hat{y}_{13} \hat{v}_{y 13}+\hat{y}_{42} \hat{v}_{y 24}\right)  \tag{9.9.20b}\\
2 \kappa_{x y}= & \frac{1}{2 A}\left(\hat{x}_{42} \hat{\omega}_{y 13}+\hat{x}_{13} \hat{\omega}_{y 24}-\hat{y}_{24} \hat{\omega}_{x 13}+\hat{y}_{31} \hat{\omega}_{x 24}\right) \\
& +\frac{2 z_{\gamma}}{A^{2}}\left(\hat{x}_{13} \hat{v}_{y 13}+\hat{x}_{42} \hat{v}_{y 24}+\hat{y}_{13} \hat{v}_{x 13}+\hat{y}_{42} \hat{v}_{x 24}\right) \tag{9.9.20c}
\end{align*}
$$

The last terms in the curvature expressions would not vanish in an arbitrary coordinate system for a rigid body rotation. However, for the coordinate system used here, the nodal velocities $\hat{v}_{x}$ and $\hat{v}_{y}$ are proportional to $z_{\gamma} \mathbf{h}$ in rigid body rotation and it can be shown that the curvatures vanish for rigid body rotation.

The hourglass strain rates are computed as in [2]; some modifications are needed to exactly satisfy the patch test. The transverse shear velocity strains are computed as described in the previous section. The stresses $\hat{\sigma}$ and the hourglass stresses $Q_{1}^{M}, Q_{2}^{M}$, $Q_{1}^{B}, Q_{2}^{B}$, and $Q_{3}^{B}$ are then computed by the constitutive equation. The nodal force expressions then emanate from the transpose of the kinematic relations.

If the corotational coordinate system $\hat{x}_{1}, \hat{x}_{2}$ is updated according to the spin as described in [2], the rate of the stress corresponds to the Green-Naghdi rate. The formulation thus requires a constitutive law which relates the Green-Naghdi rate to the corotational stretching tensor (13). Under these conditions, the formulation is valid for large membrane strains.

Shear Projection. To calculate the shear strains, a projection is made on the angular velocities

$$
\begin{equation*}
\bar{\omega}_{n}^{a}=\frac{1}{2}\left(\omega_{n I}^{a}+\omega_{n J}^{a}\right)+\frac{1}{\ell_{I J}}\left(\hat{v}_{z J}-\hat{v}_{z I}\right) \tag{9.9.21}
\end{equation*}
$$

where the superscript $a$ refers to side $a$ and the subscript $n$ refers to a component normal to side $I$; see Figure 9.?. This projection leads to a transverse shear field which is identical to the MacNeal-Wempner-Bathe-Dvorkin field. The angular velocities $\bar{\omega}_{i I}$ are obtained from $\bar{\theta}_{n}^{J}$ by

$$
\begin{align*}
& \bar{\omega}_{\hat{x} I}=\left(\mathbf{e}_{n}^{I} \cdot \mathbf{e}_{\hat{x}}\right) \bar{\omega}_{n}^{a}+\left(\mathbf{e}_{n}^{K} \cdot \mathbf{e}_{\hat{x}}\right) \bar{\omega}_{n}^{b}  \tag{9.9.24a}\\
& \bar{\omega}_{\hat{y} I}=\left(\mathbf{e}_{n}^{I} \cdot \mathbf{e}_{\hat{y}}\right) \bar{\omega}_{n}^{a}+\left(\mathbf{e}_{n}^{K} \cdot \mathbf{e}_{\hat{y}}\right) \bar{\omega}_{n}^{b} \tag{9.9.24b}
\end{align*}
$$

where $\mathbf{e}_{i}$ and $\mathbf{e}_{n}$ are unit vectors defined in Figure 9.?.


Node and side numbering


Numbering sequence

| $a$ | $J$ | $K$ |
| :---: | :---: | :---: |
| 1 | 2 | 4 |
| 2 | 2 | 1 |
| 3 | 4 | 2 |
| 4 | 1 | 3 |

Figure 9.?. Numbering scheme for shear projection.

The transverse shears at the quadrature point then are given by

$$
\begin{align*}
& 2 \hat{D}_{x z}=-\sum_{I=1}^{4} N_{I}(\xi, \eta) \bar{\omega}_{\hat{y} I}  \tag{9.9.22}\\
& 2 \hat{D}_{y z}=-\sum_{I=1}^{4} N_{I}(\xi, \eta) \bar{\omega}_{\hat{x} I} \tag{9.9.23}
\end{align*}
$$

The transverse shears do not depend on $\hat{v}_{z}$, because these velocities vanish at the quadrature point.

Evaluating the resulting forms for the transverse shear at the quadrature point, $\xi=\eta=0$, gives

$$
\begin{align*}
& \left\{\begin{array}{l}
D_{x z} \\
D_{y z}
\end{array}\right\}=\sum_{I=1}^{4}\left[\mathbf{B}_{s}\right]\left\{\begin{array}{l}
\hat{v}_{z I} \\
\hat{\omega}_{x I} \\
\hat{\omega}_{y I}
\end{array}\right\}  \tag{9.9.25}\\
& \mathbf{B}_{I}^{s}=\frac{1}{4}\left[\begin{array}{lll}
2\left(\bar{x}_{J I}-\bar{x}_{I K}\right) & \left(\hat{x}_{J I} \bar{y}_{J I}+\hat{x}_{I K} \bar{y}_{I K}\right) & -\left(\hat{x}_{J I} \bar{x}_{J I}+\hat{x}_{I K} \bar{x}_{I K}\right) \\
2\left(\bar{y}_{J I}-\bar{y}_{I K}\right) & \left(\hat{y}_{J I} \bar{x}_{J I}+\hat{y}_{I K} \bar{x}_{I K}\right) & -\left(\hat{x}_{J I} \bar{y}_{J I}+\hat{x}_{I K} \bar{y}_{I K}\right)
\end{array}\right]  \tag{9.9.26}\\
& \bar{x}_{J I}=\hat{x}_{J I} /\left(L^{J I}\right)^{2}, \quad \bar{y}_{J I}=\hat{y}_{J I} /\left(L^{J I}\right)^{2}, \quad L^{\mathrm{JI}}=\sqrt{\hat{x}_{J I}^{2}+\hat{y}_{J I}^{2}} \tag{9.9.27}
\end{align*}
$$

Table 9.2
4-Node Quadrilateral Shell Elements

| Element | Ref. | Passes <br> Patch <br> Test | Correct <br> in Twist | Speed | Robustness |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Belytschko-Tsay (BT) | [] | No | No |  | High |
| Hughes-Liu (HL) | [] | No | Yes |  | High* |
| Belytschko-Wong-Chang <br> (BWC) | [] | No | Yes |  | Moderate |
| Belytschko-Leviathan (BL) | [] | Yes | Yes |  | Moderate to <br> Low |
| YASE |  | No | No |  | Moderate |
| Full Quadrature MacNeal- <br> Wempner (Bathe-Dvorkin) |  | Yes | Yes |  | Moderate to <br> Low |

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Figure 9.2 Motion in an Euler-Bernoulli bean and a shear (Mindlin-Reissner) beam; in the Euler-Bernoulli beam, the normal plane remains plane and normal, whereas in the shear beam the normal plane remains plane but not normal. (3)

Figure 9.3 A three-node CB beam element and the underlying six-node continuum element (7)

Figure 9.4 Schematic of CB beam showing lamina, the corotational unit vectors $\hat{\mathbf{e}}_{x}$, $\hat{\mathbf{e}}_{y}$ and the director $\mathbf{p}(\xi, t)$ at the ends; note $\mathbf{p}$ usually does not coincide with $\hat{\mathbf{e}}_{y}$. (11)

Figure 9.5 A stack of quadrature points and examples of axial stress distributions for an elastic-plastic material (13)

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Figure 9.6 Reference (18)
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Table 9.2 Four-Node Quadrilateral Shell Elements

Exercise 9.?. Consider a flat plate in the x-y plane governed by the Mindlin-Reissner theory. The velocity fieldis given by

$$
\mathbf{v}=z \omega \times \mathbf{n}=z\left(\omega_{y} \mathbf{e}_{x}-\omega_{x} \mathbf{e}_{y}\right)
$$

Show that the rate-of-deformation is computed is given by

$$
\begin{aligned}
& \hat{D}_{x x}=\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{x}}+\hat{z} \frac{\partial \hat{\omega}_{y}}{\partial \hat{x}}, \quad \hat{D}_{y y}=\frac{\partial \hat{v}_{y}^{M}}{\partial \hat{y}}-\hat{z} \frac{\partial \hat{\omega}_{x}}{\partial \hat{y}} \\
& \hat{D}_{x y}=\frac{1}{2}\left(\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{y}}+\frac{\partial \hat{v}_{x}^{M}}{\partial \hat{x}}\right)+\frac{\hat{z}}{2}\left(\frac{\partial \hat{\omega}_{y}}{\partial \hat{y}}-\frac{\partial \hat{\omega}_{x}}{\partial \hat{x}}\right) \\
& \hat{D}_{x z}=\frac{1}{2}\left(\hat{\omega}_{y}+\frac{\partial \hat{v}_{z}^{M}}{\partial \hat{x}}\right), \quad \hat{D}_{y z}=\frac{1}{2}\left(-\hat{\omega}_{x}+\frac{\partial \hat{v}_{z}^{M}}{\partial \hat{y}}\right) \\
& D_{x y}=\frac{1}{2}\left(\frac{\partial v_{x}^{M}}{\partial y}+\frac{\partial v_{y}^{M}}{\partial x}\right)-\hat{z} \frac{\partial^{2} v_{z}}{\partial x \partial y}, \quad D_{x z}=D_{y z}=0
\end{aligned}
$$

Discrete momentum equation. The discrete equations for the shell are obtained via the principle of virtual power. As mentioned before, the only difference in the way the principle of virtual power is applied to a shell element is that the kinematic constraints are taken into account. We will use the same systematic procedure as before of identifying the virtual power terms by the physical effects from which they arise and then developing corresponding nodal forces. The main difference we will see is that in the shell theory nodal moments arise quite naturally, so we will treat the nodal moments separately. Boundary conditions in shells are often expressed in specialized forms, but we will first

If the angular velocity and the director are expressed in terms of shape functions, the product of shape functions will not be compatible with the reference continuum element and the result will not satisfy the reproducing conditions for linear polynomials. Therefore, the bending velocities $\mathbf{v}^{B}(\xi, \eta)$ are approximated directly.

EXAMPLE 9.?. Consider the three-node element shown, which is an application of the degenerated continuum concept to beams. The shape functions are quadratic in $\xi$. Develop the velocity field and the rate-of-deformation in the corotational coordinates. Give an expression for the nodal forces. If the nodes are placed at angles of $0^{\circ}, 5^{\circ}$, and $10^{\circ}$, what is the maximum angle between the pseudonormal $p$ and the true normal to the midline?

Expand the rate-of-deformation in $\xi$ and retain only the linear terms for an element with nodes placed along a circular arch. Compare the result with the equation.

Consider the beam element with the master nodes along the $x$-axis as shown in Figure 9.?. Develop the expression for the rate-of-deformation and compare to the Midlin-Reissner equations.


Figure 9.?.
EXERCISE. Consider the lumped mass for a rectangle.


Figure 9.?.

$$
m=\frac{1}{8} \rho_{0} a_{0} b_{0} h_{0}
$$

where $\rho_{0}, a_{0}, b_{0}$, and $h_{0}$ are the initial density and dimensions of the rectangular continuum element underlying the beam element. Using the transformation (???), develop a mass matrix and diagonalize the result with the row-sum technique.

EXERCISE. Starting with the consistent mass matrix for a rectangular continuuem element (from Przemienicki)

$$
\hat{\mathbf{M}}=[
$$

a.) develop a consistent mass using Eq. (9.3.17), i.e. $\mathbf{M}=\mathbf{T}^{T} \hat{\mathbf{M}} \mathbf{T}$ for a beam element lying along the $x$-axis as shown.


Figure 9.?
b.) develop the complete inertia term including the time-dependent term in Eq. (9.3.17).

The idea of using covariant components of velocity-strains (or strains) has already been explored in Chapter 8. It enables the assumed strain field to be tailored more precisely to arbitrarily shaped elements, independent of node numbering.

